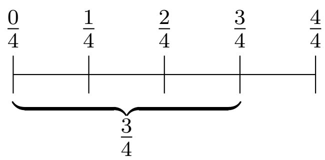
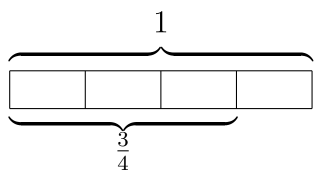
Math 3131 Prof. Pennance – Summary of Lectures XVI and XVII - Fractions

1. Let k, l be whole numbers. The *fraction* $\dfrac{k}{l}$ is the point on the number line obtained by subdividing each of the subintervals $[0, 1]$ $[1, 2]$ …etc. into l segments of equal length and counting k markers to the right.

2. Example.



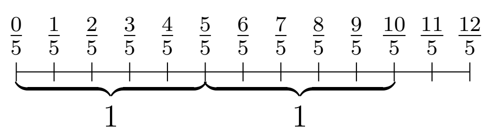
3. A bar diagram representation of $\frac{3}{4}$.



4. If $\dfrac{k}{l}$ is a fraction then k is called the *numerator* and l is called the *denominator.*

5. A fraction is called *improper* if $k \geq l$.

6. Example. The fraction $\frac{12}{5}$ is improper.



7. Geometric properties.
   (a) $\frac{1}{n}$ is the length of a part if the unit interval $[0, 1]$ is divided into n equal pieces.
   (b) $\frac{m}{n}$ is the length of a concatenation of m intervals, each of length $\frac{1}{n}$
   (c) $\frac{1}{n}$ is the area of a subrectangle if a square of size 1×1 is subdivided into n subrectangles of equal area.

   Example. $\frac{1}{6}$ is the area if a square of size 1×1 is divided into 6 equal subrectangles.

8. Simple Properties of Fractions.
   (a) $\dfrac{n}{1} = n$
   (b) $\dfrac{n}{n} = 1$
   (c) $\dfrac{np}{n} = p$
   (Since np intervals of length $\frac{1}{n}$ have a total length p)

9. Definition. Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are called *equal* if $\frac{a}{b}$ and $\frac{c}{d}$ are the same point.
   Example $\frac{1}{2} = \frac{2}{4}$.

10. *Whole-part interpretation.* $\frac{m}{n}$ is the length of a part if the interval $[0, m]$ is divided into n equal parts.

    Example If the interval [0,4] is divided into 5 equal pieces then each piece has length $\frac{4}{5}$

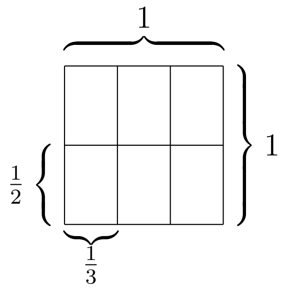
    Application. *Equal subdivision.* We wish to share 4 pizzas equally between 5 people. This is analogous to the problem of dividing the interval [0,4] into 5 equal subintervals. We can use the following algorithm.

    (a) Divide each of the unit intervals [0,1], [1,2] [2,3] and [3 ,4] into 5 subintervals of length $\frac{1}{5}$.
    (b) From each of the 4 unit intervals choose exactly one subinterval.
    (c) Concatenate the 4 chosen subintervals to form an interval of length $\frac{4}{5}$.
    (d) Repeat the above steps until no subintervals remain.

The algorithm tells us how to divide the 4 pizzas. We divide each of the 4 pizzas into 5 equal sectors. Each person chooses exactly one sector from each pizza and “concatenates” his portion. If this procedure is followed, each person will receive their fair share equal to $\frac{4}{5}$ of a pizza.

11. Definition. $\frac{a}{b} \times \frac{c}{d}$ is the area of a rectangle of sides $\frac{a}{b}$ and $\frac{c}{d}$

12. Example. By definition, $\frac{1}{2} \times \frac{1}{3}$ is the area of a rectangle of sides $\frac{1}{2}$ and $\frac{1}{3}$. A square of size 1×1 can be subdivided into 6 subrectangles each of size $\frac{1}{2} \times \frac{1}{3}$.

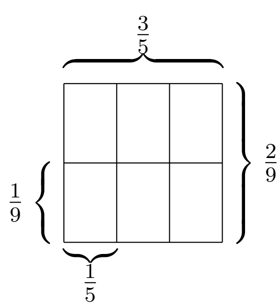


Since each subrectangle has area $\frac{1}{6}$, it follows that $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

13. By the same reasoning as in the previous example we have

$$\frac{1}{n} \times \frac{1}{m} = \frac{1}{n \times m}$$

14. Example $\frac{2}{9} \times \frac{3}{5}$ is the area of a rectangle of height $\frac{2}{9}$ and width $\frac{3}{5}$. But such a rectangle can be divided into 6 subrectangles of height $\frac{1}{9}$ and width $\frac{1}{5}$.



By the previous result each subrectangle has area $\frac{1}{9\times 5}$. Hence $\frac{2}{9} \times \frac{3}{5} = 6 \times \frac{1}{9\times 5} = \frac{2\times 3}{9\times 5}$

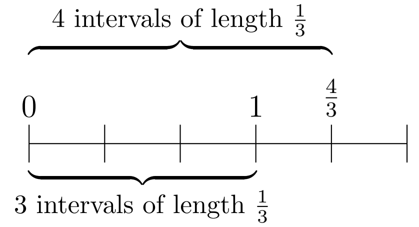
15. By similar reasoning we can show that in general

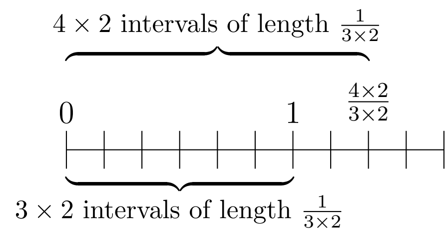
$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

16. *Cancellation property* If $m \neq 0$ then

$$\frac{k}{l} = \frac{km}{lm}$$

17. Example. $\frac{4}{3} = \frac{4\times 2}{3\times 2}$
Justification. Consider the subdivisions shown below.





18. *Addition* of fractions. Let I be any interval of length $\frac{a}{b}$ and let J be any interval of length $\frac{c}{d}$. Then the *sum* of the fractions $\frac{a}{b} + \frac{c}{d}$ is defined as the length of the concatenation of I and J.

19. The Case of Equal denominators.

$$\frac{a}{l} + \frac{c}{l} = \frac{a+c}{l}$$

Proof. Let I be an interval of length $\frac{a}{l}$ and J an interval of length $\frac{c}{l}$. Then I is a concatenation of a intervals of length $\frac{1}{l}$. Similarly J is a concatenation of c intervals of length $\frac{1}{l}$. Hence $I + J$ is a concatenation of $a+c$ intervals of length $\frac{1}{l}$ and so has total length $\frac{a+c}{l}$.

20. The Case of unequal denominators.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

Proof. We use cancellation property to reduce to the case of equal denominators.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad+bc}{bd}$$