Calculus III – Math 3153

Course Review

1. $\mathbb{R}^n$ with the operations of addition of points and scalar multiplication is a vector space.

2. Lines and line segments in $\mathbb{R}^n$

   (a) *Line* L_{PQ} where $P, Q \in \mathbb{R}^n$: $X = t_1P + t_2Q$ where $t_1 + t_2 = 1$

   (b) *Segment* S_{PQ} where $P, Q \in \mathbb{R}^n$: $X = t_1P + t_2Q$ where $t_1 + t_2 = 1$ and $0 \leq t_i$

   (c) Alternative forms of the equation of a line.

   (d) *Segment* S_{PQ}: $X = t_1P + t_2Q$ where $t_1 + t_2 = 1$ and $0 \leq t_i$

   (e) Geometric interpretation of scalar multiplication.

   (f) *Direction determined by* $v \in \mathbb{R}^2$: $[v] = \{tv : t \in \mathbb{R}\}$

   (g) *Line incident to* P *with direction* $[v]$: $\ell_{P,[v]} = P + [v]$. i.e. $\ell_{P,[v]} = \{P + tv : t \in \mathbb{R}\}$

   (h) There is exactly one line through two distinct points.

   (i) Two lines are *parallel* if they have the same direction.

3. Geometric interpretation of vector sum and difference

4. *Inner product* of points $x, y \in \mathbb{R}^n$: $< x, y >= x_1y_1 + x_2y_2 + \cdots + x_ny_n$.

5. Properties of inner product:

   (a) $< X, Y + Z >=< X, Y > + < X, Z >$ for all $X, Y, Z \in \mathbb{R}^n$.

   (b) $< X, cY >= c < X, Y >$ for all $X, Y \in \mathbb{R}^n$.

   (c) $< X, Y >=< Y, X >$ for all $X, Y \in \mathbb{R}^n$.

   (d) If $< X, Y >= 0$ for all $X \in \mathbb{R}^n$ then Y must be the zero vector.

6. *Norm* of $X \in \mathbb{R}^n$: $||X|| = \sqrt{< X, X >}$

   (a) $||X|| \geq 0$

   (b) $||X|| = 0$ implies $X = 0$

   (c) $c||X|| = |c| \cdot ||X||$

   (d) $||X + Y|| \leq ||X|| + ||Y||$

7. *Distance between two points* $X, Y \in \mathbb{R}^n$: $d(Y, Y) = ||X - Y||$

   (a) $d(X, Y) \geq 0$

(b) $d(X,Y) = 0$ implies $X = Y$

(c) $d(X,Y) = d(Y,X)$

(d) $d(X,Y) + d(Y,Z) \geq d(X,Z)$

8. $X, Y \in \mathbb{R}^n$ are *perpendicular* if $< X, Y >= 0$

9. *Projection of A on* $B(\neq 0)$: $\quad \dfrac{< A, B >}{< B, B >} B$

10. Schwartz Inequality: $< A, B >^2 \quad \leq \quad < A, A >< B, B >$

11. Geometric Interpretation of Schwarz inequality.

12. The angle between $A, B \in \mathbb{R}^n$.

13. Equations of a plane: $< n, X - P >= 0$

14. Perpendicular distance from a point A to the plane through P with unit normal u: $d = | < A - P, u > |$.

15. The two functions $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

    (a)
    i. $f(X + X', Y) = f(X,Y) + f(X',Y)$ for all $X, Y, X' \in \mathbb{R}^3$
    ii. $f(X, Y + Y') = f(X,Y) + f(X,Y')$ for all $X, Y, Y' \in \mathbb{R}^3$
    iii. $f(cX, Y) = cf(X,Y)$ for all $X, Y \in \mathbb{R}^3$ and $c \in \mathbb{R}$
    iv. $f(X, cY) = cf(X,Y)$ for all $X, Y \in \mathbb{R}^3$ and $c \in \mathbb{R}$

    (b) $f(X,X) = (0,0,0)$ for all $X \in \mathbb{R}^3$

    (c) For all $X, Y \in \mathbb{R}^3$, $f(X,Y)$ is orthogonal to both X and to Y.

    (d) If u, v are orthogonal unit vectors then $||f(u,v)|| = 1$

16. Axiomatic definition of cross product.

17. Properties of the cross product.

    (a) $(A \times B) \times C =< A, C > B - < B, C > A$

    (b) $||A \times B||^2 = ||A||^2 ||B||^2 - < A, B >^2$

    (c) $||A \times B||^2 = ||A||^2 ||B||^2 \sin^2 \theta$

    (d) $< A, B \times C >=< C, A \times B >=< B, C \times A >$

18. Some applications of the cross product.

    (a) Lorentz force: $F(v, B) = cv \times B$

    (b) Distance between skew lines: $d = | < P - Q, \dfrac{A \times B}{||A \times B||} > |$

19. The *skew product* of two vectors in $X, Y \in \mathbb{R}^2$: $X \wedge Y \;=\; X_1 Y_2 - X_2 Y_1$

20. Properties of the skew product $\wedge : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

    (a) $(c_1 v_1) \wedge (c_2 v_2) = (c_1 c_2)(v_1 \wedge v_2)$ for any scalars c_1, c_2 and any vectors v_1, v_2.

    (b) $(\cos\alpha,\ \sin\alpha) \wedge (\cos\beta,\ \sin\beta)$.

    (c) Geometric circumstance determining the sign of the skew product.

21. Curves and paths in $\mathbb{R}^n$.

    (a) Parametrization of curves.

    (b) Limits, continuity and differentiability of curves. Properties of the derivative.

    (c) Speed and acceleration.

    (d) Equations of tangent and normal lines to a differentiable curve.

    (e) The arclength of a C^1 path $\gamma : [a, b] \to R^n$: $\quad l(\gamma) = \int_a^b ||\gamma'(t)||dt$

    (f) Invariance of arclength under reparametrization by a strictly increasing C^1 function.

    (g) Reparametrization by arclength.

22. Curvature and radius of curvature:

    (a) Curvature of a path $X(s)$ parametrized by arclength: $k(s) = ||X''(s)||$

    (b) Curvature of a path $X(t)$

$$k(t) = \frac{1}{||X'(t)||} \frac{d}{dt} \left[\frac{X'(t)}{||X'(t)||} \right] = \frac{|X'(t) \times X''(t)|}{||X'(t)||^3} \ .$$

23. Formula for curvature in the case of each of the following special types of parameterized curves:

    (a) The graph of $y = f(x)$ parametrized by $X(t) = (t,\ f(t))$.

    (b) The curve $X : \mathbb{R} \to \mathbb{R}^2$ given in polar coordinates by $r = f(\theta)$. (i.e. parametrized by $X(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$.

24. Functions from $\mathbb{R}^n \to \mathbb{R}$

    (a) Level curves

    (b) Directional derivative: $D_u f(X) = \lim_{h\to 0} \dfrac{f(X + hu) - f(X)}{h}$

    (c) Partial derivatives. $D_{e_i} f(a) = \phi'(a_i)$ where $\phi(t) = f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n)$

    (d) Repeated partial derivatives

    (e) Partial differential operators

(f) If $D_1 f, D_2 f, D_1 D_2 f, D_2 D_1 f$ exist and are continuous in a region then $D_1 D_2 f = D_2 D_1 f$

(g) $g : \mathbb{R}^n \to \mathbb{R}$ is *linear* if

   i. $g(X + Y) = g(X) + g(Y)$ for all $X, Y \in \mathbb{R}^n$

   ii. $g(cX) = cg(X)$ for all $X \in \mathbb{R}^n$ and all $c \in \mathbb{R}$

(h) Differentiability of functions $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$. The derivative $Df(X)$ is a linear map i.e.

   i. $Df(X)(H+K) = Df(X)(H)+Df(X)(K)$ for all $X, Y \in \mathbb{R}^n$

   ii. $Df(X)(cH) = cDf(X)(H)$ for all $X \in \mathbb{R}^n$ and all $c \in \mathbb{R}$

(i) $Df(X)(H) =< \text{grad} f(X), H >$

(j) $D_u f(X) =< \text{grad} f(X), u >$

(k) If u is a unit vector, then $D_u f(X)$ is the projection of grad f along u.

(l) The direction of grad f is that of the maximal rate of increase of f.

(m) The norm of grad f is the rate of increase of f in the direction of maximal rate of increase.

25. Chain rule special case: $\frac{d}{dt}(f \circ X)(t) =< \text{grad} f(X(t)), X'(t) >$ where $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $X : I \subset eq\mathbb{R} \to U$ are differentiable.

26. Proof using chain rule that $D_u f(X) =< \text{grad} f(X), u >$

27. Tangent planes and normal lines of a regular surfaces.

28. Vector fields in $\mathbb{R}^n$

   (a) Potential functions and conservative fields.

   (b) Uniquness up to a constant of the potential in a path connected region.

   (c) Necessary conditions for the existence of a potential for $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. Let $F = (f, g)$. If $D_1 f, D_2 f, D_1 g, D_2 g$ are continous and F is conservative then $D_2 f = D_1 g$.

   (d) Let f be continuous on $[a, b] \times [c, d]$. If $D_2 f$ exists and is continuous on $[a, b] \times [c, d]$, then $\frac{d}{dy} \int_a^b f(x, y)dx = \int_a^b D_2 f(x, y)dx$

   (e) Paths and curves in $\mathbb{R}^n$

   (f) Integral of a continuous vector field F along a C^1 path $\gamma : [a, b] \to \mathbb{R}^3$: $\int_\gamma f = \int_a^b < F(\gamma(t)), \gamma'(t)dt$

   (g) Integral of a continuous vector field F along a piecewise C^1 path $\gamma : [a, b] \to \mathbb{R}^3$: $\int_\gamma f = \int_a^b < F(\gamma(t)), \gamma'(t) > dt$

(h) Path integrals and conservative fields.

(i) Application: Work done in inverse square fields.

(j) Conservation of energy in a conservative field.

(k) Equivalence of the following statements for a continuous vector field F defined on an open path connected set U

   i. F has a potential function on U

   ii. $\int_\gamma F = 0$ for all closed, piecewise C^1 paths γ.

   iii. $\int_{[P,Q],\gamma} F$ is independent of path.

29. (a) Taylor's theorem in one and higher dimensions.

    (b) Properties of the remainder

30. Optimization

    (a) Extrema of real valued functions

    (b) Quadratic forms in 2 variables.

    (c) If $g(h) = \frac{1}{2}(h_1, h_2)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ then g is positive definite if and only if $ac - b^2 > 0$ and $a > 0$ and negative definite if and only if $ac - b^2 > 0$ and $a < 0$.

    (d) The Hessian $\mathcal{H}_P(H) = \dfrac{< H, \nabla >^2}{2!} f(P)$

    (e) Sufficient conditions for local maxima and minima of a class C^3 function $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ at a critical point where U is open.

    (f) Lagrange multipliers

31. Multiple integrals

    (a) The double integral of a continuous function over a closed rectangle and over a closed bounded region $\Omega \subset eq\mathbb{R}^2$ and properties.

    (b) The mean value theorem.

    (c) Iteraled integrals

    (d) Fubini's theorem

    (e) Changes of variable

    (f) Polar coordinates $X(r, \theta) = (r\cos\theta, r\sin\theta)$.

    (g) The triple integral over special regions and reduction to iterated integrals.

    (h) Cylindrical and spherical coordinates.

    (i) Transformation of volume elements under cylindrical and spherical coordinates.

(j) The determinant of a 2×2 matrix viewed as the unique bilinear map $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ satisfying

   i. g is bilinear.

   ii. $g(v, v) = 0$ for all $v \in R^2$

   iii. $g(e_1, e_2) = 1$

(k) Determinants in general.

(l) Signed volume and determinants.

(m) The derivative of a function $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$.

(n) The Jacobian matrix.

(o) The general change of variables formula.

(p) Inverse mapping and implicit function theorems.

32. Surfaces

(a) The image S of a map $X : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ where D a bounded region, bounded by a finite number of smooth curves is called a *parametrized surface*. S is a C^1 parametrized surface if X is C^1 and *smooth* if $X_u \times X_v \neq 0$ for all $(u, v) \in D$.

(b) Integrals over smooth parametrized surfaces:

   i. If $S = \text{im}X$ is smooth and $X : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ injective, then the *area* of S is defined by: $A(S) = \int_D ||X_u \times X_v|| dudv$

   ii. Integral of a scalar function over a smooth parametrized surface: $\int_S f d\sigma = \int_D f(X(u, v)) ||X_u \times X_v|| dudv$.

   iii. Integral of a vector field over a smooth parametrized surface: $\int_S F \cdot d\sigma = \int_D < F(X(u, v)), X_u \times X_v > dudv$

33. Green's theorem.

(a) Green's theorem: *Let Ω be a Jordan region with piecewise-smooth boundary $\partial\Omega$. If P and Q are scalar fields continuously differentiable on an open set that contains Ω then*

$$\int_{\partial\Omega} P(x, y)dx + Q(x, y)dy = \int\int_{\Omega} \left[\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y} \right] dxdy$$

with integration over $\partial\Omega$ taken in counterclockwise direction.

(b) Proof of Greens theorem in special cases and extension to anular regions.

34. Gauss' Divergence Theorem:

   (a) Gauss' Divergence Theorem: *Let U be a region in $\mathbb{R}^3$ whose boundary is a closed surface which is smooth except for a finite number of smooth curves. Let F be a C^1 vector field on an open set containing U and S. Let n be the unit outward normal vector field to S. Then*

$$\int\int_S F \cdot n d\sigma = \int\int_v\int \text{div} F dV$$

   i.e. The flux of a vector field F out of a closed surface S equals the integral of the divergence of that vector field over the volume enclosed by the surface.

   (b) Proof of Gauss' divergence theorem for a rectangular box.

   (c) Interpretation of divergence of a current density J at P as the rate of change of mass per unit volume P

   (d) Applications to electromagnetism and fluid flow.

35. Stokes' theorem.

   (a) Stokes' theorem. Let S be a smooth orientable surface in $\mathbb{R}^3$, bounded by a closed curve C. Let n be the unit normal vector field on S and Let C have the positive orientation induced by n. Let F be a C^1 vector field in an open set containing the surface S and its boundary. Then

$$\int\int_S (\text{curl} F) \cdot n d\sigma = \int_C F \cdot dC$$

   (b) Proof of Stokes theorem for the special case where S is graph of a C^2 function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and D is a Jordan region (so that Green's theorem applies).