

1. Let F and G be statements. By definition, the statement “ F implies G ” (also written $F \Rightarrow G$) is true if G is true whenever F is.
2. Corollary. $F \Rightarrow G$ is true if and only if it cannot happen that F be true and at the same time G be false.
3. Fire is a chemical reaction involving oxygen. Thus it cannot happen that there is fire without oxygen. Hence the implication “Fire implies the presence of oxygen” is a true statement.
4. It can (and fortunately usually does) happen that presence of oxygen can occur with no fire and so by corollary [2] above the implication “oxygen implies fire” is false. Thus order is important. $F \Rightarrow G$ and $G \Rightarrow F$ are NOT equivalent.
5. Alternative expressions of the truth of the implication “Fire implies the presence of oxygen”
 - (a) Fire **implies** the presence of oxygen.
 - (b) **If** there is fire **then** there is presence of oxygen
 - (c) There is **no** fire **without** oxygen.
 - (d) **Whenever** there is fire there is oxygen.
 - (e) Fire is a **sufficient** condition for presence of oxygen.
 - (f) There is fire **only if** there is oxygen.
 - (g) Oxygen is **necessary** for fire.
 - (h) There is oxygen **whenever** there is fire.
 - (i) There is oxygen **if** there is fire.
6. Expressions equivalent to $F \Rightarrow G$
 - (a) F **implies** G .
 - (b) **If** F **then** G .
 - (c) F is **not true unless** G is true.
 - (d) **Whenever** F is true G is true.
 - (e) F is **sufficient** for G .
 - (f) F **only if** G .
 - (g) G is **necessary** for F .
 - (h) G **whenever** F .
 - (i) G **if** F .
7. Words such as “provided”, “since”, “therefore”, and “unless” also express implications:

$$\begin{aligned}
 F \text{ provided } G &\equiv G \Rightarrow F \\
 F \text{ since } G &\equiv G \Rightarrow F \\
 F \text{ therefore } G &\equiv F \Rightarrow G \\
 F \text{ unless } G &\equiv \neg G \Rightarrow F
 \end{aligned}$$

Note that “unless” is equivalent to “if not”.
8. Definition. The *contrapositive* of the implication $F \Rightarrow G$ is the implication $\neg G \Rightarrow \neg F$.
9. The contrapositive $\neg G \Rightarrow \neg F$ is true if and only if it cannot happen that $\neg G$ is true and $\neg F$ is false. This is the same as saying that it cannot happen that F is true and G is false. But this is the same as saying that $F \Rightarrow G$ is true. Therefore the statements $\neg G \Rightarrow \neg F$ and $F \Rightarrow G$ are equivalent.
10. Example. “If there is no oxygen, then there is no fire” is the contrapositive of, and hence equivalent to, “if there is fire then there is oxygen.”

11. If an implication $F \rightarrow G$ is false we say that F is not a sufficient condition for G or that G is not a necessary condition for F . “oxygen implies fire”
12. Example. The following statements all express the falsehood of the statement
- (a) “Presence of oxygen is not a sufficient condition for fire”.
- (b) “Fire is not a necessary condition for presence of oxygen”.

Exercises

- Let F be the statement “It is raining” and G is the statement “I use my umbrella”. Write the following implications symbolically.
 - “I use my umbrella whenever it rains”.
 - “When it rains I use my umbrella”.
 - “I use my umbrella if it rains”.
 - It rains only if I use my umbrella.
 - I use my umbrella only if it rains.
 - If it rains it is necessary that I use my umbrella.
 - Rain is sufficient for me to use my umbrella.
- Write 10 different statements which are equivalent to “There is no smoke without fire”.
- Suppose that an implication $F \Rightarrow G$ is true. If G is false, what can be deduced about the truth of F ? Explain.
- Let n be a natural number.
 - Write the contrapositive of the statement “If 2 is a factor of n^2 then 2 is a factor of n ”.
 - Prove the contrapositive of the statement in part (a).