Math 4032 – Prof. Pennance –Group Axioms

1. Let A be a set. A *binary operation* on A is a function $\otimes : A \times A \to A$.

2. If $\otimes$ is a binary operation we write $a \otimes a'$ instead of $\otimes(a, a')$.

3. A binary operation $\otimes$ on a set A is *associative* if $a \otimes (a' \otimes a'') = (a \otimes a') \otimes a''$ for all $a, a', a'' \in A$.

4. A *grupoid* is a pair $(A, \otimes)$ where A is a nonempty set and $\otimes$ is a binary operation on A.

5. Let $(A, \otimes)$ be a grupoid. An element $e \in A$ is called an *identity* or *neutral element* if $e \otimes a = a \otimes e = a$ for all $a \in A$.

6. Identity elements are unique.

7. Let $(A, \otimes)$ be a grupoid with an identity element e and Let $a \in A$. An element $a' \in A$ is called an *inverse* of a if $a \otimes a' = a' \otimes a = e$.

8. A grupoid $(G, \otimes)$ is called:
   (a) A *semigroup* if $\otimes$ is associative.
   (b) A *monoid* if
      i. $\otimes$ is associative.
      ii. G has an identity element.
   (c) A *group* if
      i. $\otimes$ is associative.
      ii. G has an identity element.
      iii. Every $g \in G$ has an inverse.

9. A group $(G, \oplus)$ is called *abelian* if $\oplus$ is commutative.

10. In a monoid inverse elements (if they exist) are unique.

11. Examples of semigroups:
   (a) $(\mathbb{N} - 0, +)$
   (b) Any monoid

12. Examples of monoids:
   (a) $(\mathbb{N}, +)$.
   (b) $(2^E, \cap)$ where E is a set.
   (c) Any group.

13. Examples of groups:
   (a) $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$.
   (b) $(\mathbb{Q} - 0, \cdot)$, $(\mathbb{R} - 0, \cdot)$, $(\mathbb{C} - 0, \cdot)$.
   (c) $(\{1, -1\}, \cdot)$.
   (d) The permutation group $S_N = (\{1, 2, \ldots, N\}!, \circ)$.
   (e) $(\mathbb{R}^2, +)$ where $(a, b) + (a', b') = (a + a', b + b')$.
   (f) The product of two groups.
   (g) The set of solutions of the differential equation of a spring $X'' + X = 0$.
   (h) $(2^E, \Delta)$ where E is a set and Δ is symmetric difference.
   (i) The roots of unity $(\{z \in \mathbb{C} : z^n = 1\}, \cdot)$.
   (j) The set $GL_2(\mathbb{R})$ of invertible 2×2 real matrices under matrix multiplication.
   (k) The symmetry group of a set $S \subseteq \mathbb{R}^2$.
   (l) Groups of orders 1, 2, and 3.

$*$	e
e	e

$*$	e	a
e	e	a
a	a	e

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

14. The order of a finite group.

15. The order of an element in a group.