Math 4032   Prof. Pennance – Summary of Elementary Group Properties

1. Let $(G, *)$ be a group and $a, b \in G$.
   (a) $(a^{-1})^{-1} = a$.
   (b) $(ab)^{-1} = b^{-1}a^{-1}$.
   (c) Cancellation property
   If $a * c = b * c$ then $a = b$.

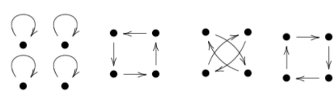
2. Corollaries of cancellation property:
   (a) Latin square property.
   (b) Let $g \in G$, then the function $\lambda_g : G \to G$ defined by $\lambda_g(x) = g * x$ for all $x \in G$ is a bijection.

3. Recursive definition of exponential function $n \mapsto a^n$ where a is an element of a group G and $n \in \mathbb{Z}$.

4. Properties of the exponential function:
   (a) $a^n * a^m = a^{n+m}$
   (b) $a^n * a^{-m} = a^{n-m}$
   (c) $(a^n)^m = a^{nm}$

5. A group G is *cyclic* if there exists an element $a \in G$ (called a *generator*) such that every element of G has the form a^n for some $n \in \mathbb{Z}$.

6. Examples of cyclic groups:
   (a) All groups of orders 1, 2 or 3.
   (b) The group of permutation (under composition)



   (c) $(\mathbb{Z}, +)$

7. Abelian groups.
   (a) Additive notation for abelian groups.
   (b) If $a^2 = e$ for all $a \in G$ then G is abelian.
   (c) Cyclic groups are abelian.
   (d) If G is abelian then $(a*b)^n = a^n b^n$.

8. The cyclic group of order 4 and table.

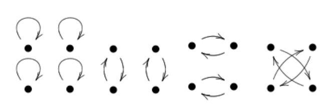
$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	e

   (a) $Z_4 = (\{0, 1, 2, 3\}, +_4)$
   (b) $(\{2, 4, 6, 8\}, \times_{10})$
   (c) Rotation group of the square.
   (d) $\{z \in \mathbb{C} : z^4 = 1\}$ under complex multiplication.

9. The Klein group.
   If a group $K = \{e, a, b, c\}$ of order 4 has no element of order 4 then necessarily K has multiplication table

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	a	b	e

   (a) The group of permutations



   (b) The symmetry group of a rectangle.
   (c) $(\{0, 1\}^2, (+_2, +_2))$
   (d) $(2^A, \Delta)$ where $A = \{a, b\}$.
   (e) The group $\{\pm \text{id}, \pm \frac{1}{\text{id}}\}$ of real valued functions under composition.

10. The cyclic group of order 5 and table.

11. The symmetry group D_n of a regular n-gon.