Math 4032 Prof. Pennance – Summary of Lecture on Homomorphisms

1. Let $(A, \odot)$ and $(B, \star)$ be groups. A function $h : A \to B$ is a *group homomorphism* if

$$h(x \odot y) = h(x) * h(y)$$

for all $x, y \in A$.

Interpretation:

$\odot$	$\dots$	y	$\dots$
$\vdots$			
x	$\dots$	$x \odot y$	$\dots$
$\vdots$		$\vdots$	

$\downarrow$

$\star$	$\dots$	h(y)	$\dots$
$\vdots$			
h(x)	$\dots$	$h(x \odot y)$	$\dots$
$\vdots$		$\vdots$	

Note: Unless h is a bijection, the tables above may be of different sizes.

2. $h : A \to B$ is a group homomorphism if and only if the following diagram commutes.

$$\begin{array}{ccc} A \times A & \xrightarrow{h \times h} & B \times B \\ \odot \downarrow & & \downarrow \star \\ A & \xrightarrow[h]{} & B \end{array}$$

3. Examples of homomorphisms.

   (a) Let G and H be groups. The function $\phi : G \to H$ given by $\phi(g) = e_H$ for all $g \in G$ is called the trivial homomorphism.

   (b) Let G be a group, and let $a \in G$. The exponential function $f : \mathbb{Z} \to G$ given by $f(n) = a^n$ is a homomorphism.

   (c) More generally, let C be an infinite cyclic group with generator a. Let G be any group. Define $f : C \to G$ by $f(a^m) = g^m$. Then f is a group homomorphism. In the case of a finite cyclic group, f defines a homomorphism provided that $o(g)$ divides $o(a)$.

   (d) The projection map $\pi : \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x, y) = x$.

   (e) The absolute value function

$$|\ | : (\mathbb{R} - 0, \cdot) \to (\mathbb{R}^+, \cdot)$$

   (f) The determinant function from $GL_2(\mathbb{R}, \cdot) \to (\mathbb{R} - 0, \cdot)$

   (g) $h : (\mathbb{Z}, +) \to (5Z, +)$ given by $h(n) = 5n$

   (h) Let G be the set of 2×2 real matrices of the form

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

   Then G is a group under matrix multiplication and

$$\phi : (\mathbb{R}, +) \to (G, \cdot)$$

   given by $\phi(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is a homomorphism. (In elasticity theory, elements of G describe shear).

4. Let $f : G \to H$ be a group homomorphism. Then

   (a) $f(e_G) = e_h$

   (b) $f(a^{-1}) = (f(a))^{-1}$ for any $a \in G$

   (c) $f(a^n) = (f(a))^n$

   (d) For any $a \in G$ the order of $f(a)$ divides the order of a.