Math 4032 Prof. Pennance – Summary of Lecture on Isomorphisms

1. Let $f : (G, \odot) \to (H, *)$ be a group homomorphism. The kernel of f, is the set

$$\ker f = \{g \in G : f(g) = e_H\}$$

2. Let $f : (G, \odot) \to (H, *)$ be a group homomorphism.

   (a) $\ker f \leq G$

   (b) f is injective if and only if $\ker f = \{e_G\}$.

   (c) $\mathrm{im} f \leq H$

   (d) If $G' \leq G$ then $f[g'] \leq H$

   (e) If $H' \leq H$ then $f^{-1}[H'] \leq G$

3. An homomorphism of groups which is also bijective is called an *isomorphism*. Two groups G and H are called isomorhpic (written $G \sim H$) if there is an isomorphism from G to H.

   Examples of isomorphisms:

   (a) The identity function on any group.

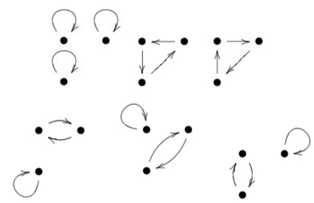
   (b) $\log : (\mathbb{R}^+, \cdot) \to (\mathbb{R}, +)$ where

$$\log x = \int_1^x \frac{1}{t} dt$$

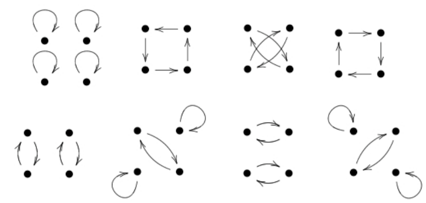
   Corollaries: $\log 1 = 0$ and $\log x^{-1} = -\log x$

   (c) Congugation $(\mathbb{C}, +) \to (\mathbb{C}, +)$ defined by $x + iy \mapsto x - iy$.

   (d) The Dihedral group D_3 is isomorphic to the permutation group



   (e) The Dihedral group D_4 is isomorphic to



   (f) Let $\chi : 2^{\mathbb{N}_8} \to \mathcal{B}_8$ be the usual coding of subsets of $\mathbb{N}_8$ by binary numbers e.g. $\{1, 2, 4, 5, 6, 8\} \overset{\chi}{\longmapsto} 11011101$
   Notice $\mathcal{B}_8$ is a group under the addition modulo 2 operation
   $XOR = (+_2, +_2, +_2, +_2, +_2, +_2, +_2, +_2)$
   and $\chi : (2^{\mathbb{N}_8}, \Delta) \to (\mathcal{B}_8, XOR)$ is an isomorphism.

$$\begin{array}{ccc} 2^{\mathbb{N}_8} \times 2^{\mathbb{N}_8} & \xrightarrow{\chi \times \chi} & \mathcal{B}_8 \times \mathcal{B}_8 \\ \Delta \downarrow & & \downarrow XOR \\ 2^{\mathbb{N}_8} & \xrightarrow[\chi]{} & \mathcal{B}_8 \end{array}$$

4. (Categorical property) If f : $G \to H$ and $g : H \to K$ are group homomorphisms, then so is the composition $g \circ f : G \to K$.

5. The inverse function of an isomorphism is an isomorphism.
   Corollary: $e^{(a+b)} = e^a e^b$