**Math 4032 Prof. Pennance – Summary of Lecture on Lagrange's Theorem**

1. Let $G$ be a group, $H$ a subgroup of $G$ and $a \in G$. The *right coset* of $H$ determined by $a$ is the set

$$Ha = \{ha \in G : h \in H\}$$

.

2. Let $G$ be a group and $H$ a subgroup of $G$. For any $a, b \in G$ let $a \sim b$ if and only if $ab^{-1} \in H$. Then $\sim$ is an equivalence relation on $H$. Moreover, for each $a \in G$ the equivalence class of $a$ is the right coset determined by $a$. i.e., $[a] = Ha$.

3. Corollary. Let $G$ be a group and $H$ a subgroup of $G$. Then the right cosets of $H$ partition $G$.

4. More properties of cosets.
   (a) $Ha = Hb \Leftrightarrow a \in Hb \Leftrightarrow a \sim b$.
   (b) $Ha = H \Leftrightarrow a \in H$.

5. Let $G$ be a finite group and $H$ a subgroup of $G$ and $a \in G$. Then $|Ha| = |H|$ (and so all right cosets have the same cardinality).

6. (Lagrange's Theorem). Let $G$ be a finite group and $H$ a subgroup of $G$. Then $|H|$ is a factor of $|G|$.

7. Some Corollaries
   (a) Let $G$ be a finite group and $a \in G$. Then $O(a)|O(G)$
   (b) Let $G$ be a finite group and $a \in G$. Then $a^{O(G)} = e_G$.
   (c) Let $G$ be a non-cyclic group of order 4. Then every element of $G$ except the identity has order 2.

8. Let $G$ be a finite group. The following are equivalent.
   (a) $|G|$ is prime.
   (b) There is no subgroup $H$ of $G$ such that $e < H < G$ (i.e., all subgroups are trivial).

   Proof. $(a) \Rightarrow (b)$ follows immediately from Lagrange's theorem. Let (b) is true. Then every non zero element $a \in G$ must have order $|G|$. Suppose to the contrary that $|G|$ is not prime. Then $|G| = m \cdot n$ where $1 < m, n < |G|$. But then

$$O(a^n) = \frac{O(a)}{\text{gcf}(O(a), m)} = \frac{O(a)}{n}$$

   which is a contradiction. Hence $G$ must be prime.

9. Let $G$ be a group of order 6. Then $G$ is isomorphic to either the cyclic group $C_6$ or to the permutation group $S_3$.