1. Aristotle (384-322 BC) studied arguments which consisted of two or more statements called *premises* and a statement called the *conclusion*. We will be interested mainly in the special case of arguments with two premises F and G and one conclusion C. Such arguments are often presented in the format:

$$\begin{array}{l} F \\ G \\ \hline C \end{array}$$

2. An argument with two premises F and G and a conclusion C is called *valid* if the truth of the conjunction of the premises $F \wedge G$ is a sufficient condition for the truth of the conclusion C (i.e., if the implication $F \wedge G \Rightarrow C$ is true).

3. To check the validity of an argument, it suffices to show that the conclusion is true whenever the conjunction of the premises is true.

4. Example. Consider an argument of the form:

$$\begin{array}{l} F \\ F \Rightarrow G \\ \hline G \end{array}$$

   The two premises of this argument are F and $F \Rightarrow G$ and the conclusion is G. To show that the argument is valid, suppose that both premises are true (i.e. suppose that F is true and that the implication $F \Rightarrow G$ is also true). The truth of the latter implication means that it CANNOT happen that F be true and at the same time G be false. Since F is true by hypothesis, it can only be that G is true. Thus, the conclusion of the argument is true whenever both premises are true. Therefore, the argument is valid.

5. Validity of an argument does not depend on the nature of the statements involved but only the form of the argument. Any statements which we substitute for F and G in the above form will produce a valid argument.

6. Example. Let F be the statement, "Rex is a dog", and G the statement, "Rex has fleas" then the argument

$$\begin{array}{l} \text{Rex is a dog} \\ \text{If Rex is a dog then Rex has fleas} \\ \hline \text{Rex has fleas} \end{array}$$

   has the form discussed in [4] above which we already have shown to be valid.

7. Alternatively, we can show the argument in [6] to be valid by verifying that the propositional formula

$$(p \wedge (p \Rightarrow q)) \Rightarrow q$$

   is a tautology (exercise).

8. The validity of an argument does not depend on the truth or falsehood of the premises. The argument in the previous example is valid even if the premise Rex is a dog is false.

9. The argument

$$\begin{array}{l} \text{If } n > 2 \text{ then } n^2 > 4 \\ n^2 > 4 \\ \hline n > 2 \end{array}$$

   has the form

$$\begin{array}{l} F \Rightarrow G \\ G \\ \hline F \end{array}$$

An argument of this form is invalid since both premises can be true and at the same time the conclusion F false.

This is the case, for example, when F is false and G is true. In this case $n = -3$ provides a counter example.

Exercises

1. Show that each of the following arguments are valid.

(a) $\begin{array}{l} F \Rightarrow G \\ F \\ \hline G \end{array}$

(b) $\begin{array}{l} \neg(F \wedge G) \\ \hline \neg F \vee \neg G \end{array}$

(c) $\begin{array}{l} \neg(F \vee G \\ \hline \neg F \wedge \neg G \end{array}$

(d) $\begin{array}{l} \neg G \Rightarrow \neg F \\ \hline F \Rightarrow G \end{array}$

(e) $\begin{array}{l} F \Rightarrow G \\ G \Rightarrow H \\ \hline F \Rightarrow H \end{array}$

(f) $\begin{array}{l} F \Rightarrow (G \wedge \neg G) \\ \hline \neg F \end{array}$

2. Write the following arguments in symbolic form and check for validity.

(a) If it rains I use an umbrella
I use an umbrella
____
It is raining

(b) If it rains I use an umbrella
I do not use an umbrella
____
It is not raining

(c) If it rains I use an umbrella
It is not raining
____
I am not using an umbrella

3. Which of the following arguments are valid? Prove your answers.

(a) $\begin{array}{l} F \Rightarrow G \\ \neg G \\ \hline \neg F \end{array}$

(b) $\begin{array}{l} F \\ G \Rightarrow F \\ \hline F \end{array}$

(c) $\begin{array}{l} \neg F \\ F \Rightarrow G \\ \hline \neg G \end{array}$

(d) $\begin{array}{l} \neg F \\ F \Rightarrow G \\ \hline G \end{array}$

(e) $\begin{array}{l} F \\ F \Rightarrow G \\ \hline \neg G \end{array}$

(f) $\begin{array}{l} F \\ F \Rightarrow G \\ \hline G \end{array}$