Math 3023 – Philip Pennance http://pennance.us/ – Set theory Lite

Basic to the study of all mathematics is the concept of *set*. The reason why set theory is important to mathematics is that the numbers are constructed by using sets and so numbers like 1, 2, 3 turn out to be sets.

In set theory we do not define the terms set and element– it is impossible to define everything without being circular. However, it is helpful for our intuition to think of a set as a collection of individual elements such as potatoes. A concept such as *set*, which we do not define, is called a *primitive*. Every theory, mathematical and scientific contains primitive concepts. For example in classical thermodynamics, *time* is a primitive and both *temperature* and *heating* are primitive functions of time. In Euclidean geometry the concepts of point and line are usually taken as primitives. In any theory, it is essential to know which concepts are primitive and which have been given in a definition.

Set theory like any other theory requires a language which in turn requires an alphabet. The alphabet of set theory contains the following types of letter.

1. Variables: $a, b, c, \ldots A, B, C, \ldots$. In this course, small letters will represent elements and capital letters will stand for sets.
2. Logical symbols: $\wedge$ (and) , $\vee$ (or) , $\neg$ (not) , $\Rightarrow$ (implies), $\Leftrightarrow$ (if an only if).
3. Quantifiers: $\forall$ (for all), $\exists$ (for some).
4. Relational Symbols: $=$ (equals), $\in$ (belongs to).

We also use parentheses as necessary.

If an element a belongs to a set A we write $a \in A$. Two sets A and B are called *equal* (written $A = B$) if they have the same elements. Sentences of the forms $a \in A$ and $A = B$ are called *atomic*. If F and G are sentences then the following sentences can be formed.

1. $F \wedge G$
2. $F \vee G$
3. $F \Rightarrow G$
4. $F \Leftrightarrow G$
5. $\neg F$
6. $\forall v F$
7. $\exists v F$

The symbol $\emptyset$ will be used to represent the set with no elements –also known as the *empty set*. We can specify sets by listing their elements. For example $\{a, b, c\}$ denotes the set whose elements are a, b, and c. When a set is defined by giving a list of its elements we say that it is defined by *extension*.

We can also define sets by means of a predicate. Let E be a set and $p(x)$ a predicate valid for the set E. We assume that

$$\{x \in E : p(x)\}$$

is a set. The elements of this set are precisely those elements of the set E for which the predicate is true. This is completely analogous to the Aristotelian concept of definition. The set E plays the role of genus and the predicate $p(x)$ the differentia. For example, given the set of integers $\mathbb{Z}$ we can write the subset of even integers:

$$\{n \in \mathbb{Z} : n \text{ is twice an integer}\}$$

which is read "The set of integers n such that n is twice an integer". This method of defining a set is called *specification.* In Aristotelian terminology, the species *even* is defined by specifying its genus (the integers) and the differentia "n is twice an integer".

Let A and B be sets. We say that A is a *subset* of B and write $A \subseteq B$ if every element of A is an element of B. If A is not a subset of B we write $A \not\subseteq B$. For example $\{1,2,3\} \subseteq \{1,2,3,4\}$ but $\{1,2,3,4\} \not\subseteq \{1,2,3\}$. Notice that in order for two sets A and B to be equal both $A \subseteq B$ and $B \subseteq A$ must be true.

Exercises

In the following exercises $\mathbb{R}$ denotes the sets of real numbers and $\mathbb{Q}$ the set of rational numbers. $\mathbb{R}^2$ denotes the set of all ordered pairs of real numbers. Similarly $\mathbb{N}^2$ denotes the set of all ordered pairs of natural numbers.

1. We know that two sets are *equal* if they have the same elements. Are two humans *equal* if they have the same ancestors? Are the sets $\{3\}$ and $\{3,3\}$ equal? Explain.

2. For any two elements a and b let (a,b) denote the set $\{\{a\},\{a,b\}\}$.

   (a) Find the sets $(1,2)$ and $(2,1)$. Are these sets equal?
   (b) How many elements does the set $(1,1)$ have?

3. Express by specification the following sets:

   (a) $\{1,2,3,4,5\}$
   (b) $\{2,3,5,7,11,13,17,19,23,29,31\}$
   (c) $\{3,6,9,12,\ldots,999\}$
   (d) $\{3,8,15,24,35,48,63\}$
   (e) $\{64,128,256,512,1024\}$

4. Express by extension the following sets:

   (a) $\{x : x \text{ is a capital city of one of the countries of Hispaniola}\}$
   (b) $\{n \in \mathbb{N} : 3 \leq n \leq 6\}$
   (c) $\{x \in \mathbb{Q} : 6x^2 = 2 - x\}$
   (d) $\{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = 0\}$
   (e) $\{(x,y) \in \mathbb{R}^2 : x + y = 1 \text{ and } x - y = 2\}$
   (f) $\{(n,m) \in \mathbb{N}^2 : 2m + 4n = 13\}$
   (g) $\{(n,m) \in \mathbb{N}^2 : n^2 = 2m^2\}$