Math 3023 – (Philip Pennance http://pennance.us/) – – Functions I

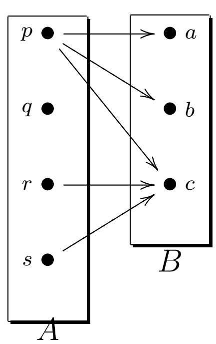
1. A triple (A, B, R) is called *relation* or *table* if
   (a) A is a set and
   (b) B is a set and
   (c) R is a set of ordered pairs such that every ordered pair (a, b) in R has its tail a in A and its head b in B (and so $R \subseteq A \times B$).

   The sets A and B in the above definition are called the *domain* and the *codomain* of the relation respectively. The set of ordered pairs R is called the *graph* of the relation.

2. Example Let $A = \{p, q, r, s\}$, $B = \{a, b, c\}$ and

   $$R = \{(p, a), (p, b), (p, c), (r, c), (s, c)\}.$$

   Since $R \subseteq A \times B$, it follows that the triple $T = (A, B, R)$ is a relation. The relation T determines a (directed) (A, B)-bipartite graph which is drawn below.



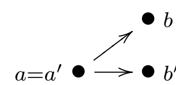
   The ordered pairs in R are drawn as directed edges (arrows) and the elements of A and B as vertices (points).

   The *degree* of a vertex v is defined to be the number of edges incident to v and is denoted $d(v)$. In the diagram $d(p) = 3$, $d(q) = 0$, $d(c) = 3$ etc.

3. A relation $f = (A, B, R)$ is called a *function* or a *function from A to B* if in the corresponding bipartite graph, every vertex in the domain has degree exactly one.

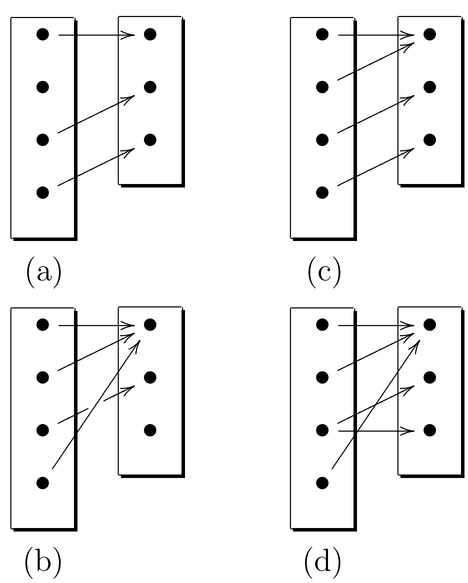
4. Clearly, a relation $f = (A, B, R)$ is a function if and only if the following two conditions both hold:
   (a) For every $a \in A$, there exists $b \in B$ such that $(a, b) \in R$
   (b) If $(a, b) \in R$ and $(a', b') \in R$ and $a = a'$ then $b = b'$.

   Condition (1) guarantees that the degree of every vertex in the domain is at least 1. Condition (2) ensures that the degree of every vertex in the domain is at most one thereby prohibiting configurations such as:



   The conjunction of the two conditions is equivalent to the statement that every vertex in the **domain** has degree exactly 1.

5. Example. Consider the relations depicted below.



Only the drawings (b) and (c) depict functions. The relation corresponding to drawing (a) is not a function since the domain contains a vertex of degree less than 1. The graph in (d) is not a function since the domain posseses a vertex of degree more than 1. Notice that the property of being a function does not depend on the degrees of the vertices in the codomain.

6. Functional Notation If $f = (A, B, R)$ is a function from A to B we will write $f : A \to B$. The domain A will be denoted D_f, the codomain B by C_f and the graph R by G_f. The **unique** vertex adjacent to a in the bipartite graph of f is denoted by $f(a)$. With this notation

$$G_f = \{(x, f(x)) | x \in A\}$$

The following notations and terminologies are all synonymous:

(a) $(a, b) \in R$.

(b) $(a, b) \in G_f$.

(c) (a, b) is an edge in the bipartite graph of f.

(d) $f(a) = b$.

(e) $f : a \mapsto b$.

(f) $a \overset{f}{\mapsto} b$

(g) f maps a to b.

(h) f associates b with a.

(i) b is the image of a under the function f.

7. Example. The triple

$$f = (\mathbb{R}, \mathbb{R}, G_f)$$

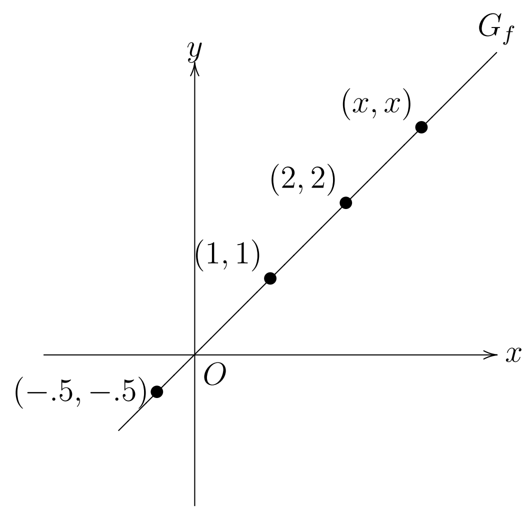
where $G_f = \{(x, x) : x \in \mathbb{R}\}$ is a function with domain $\mathbb{R}$ and codomain $\mathbb{R}$. Its graph consisting of all pairs of real numbers of the form (x, x). This function is called the identity function on $\mathbb{R}$. In our alternative notations the graph can be specified by the expression

$$f(x) = x, \quad x \in \mathbb{R}.$$

Since the graph G_f is a subset of the cartesian plane, it can be sketched in the usual manner (see figure below). Geometrically, the graph is just the line with equation $y = x$. Of course, the definition does not depend on letters. We could just as well specify the graph by

$$f(\zeta) = \zeta, \quad \zeta \in \mathbb{R}$$

if for no other reason than to force students to learn the greek alphabet.



8. Examples The following define graphs of functions with domain $\mathbb{R}$.

(a) $f(x) = x + c$ (translation by c)

(b) $g(x) = -x$ (reflection in the origin)

(c) $h(x) = cx$ where $c > 0$ (dilation)

(d) $f(x) = mx + b$ (affine function).

In each case (as you should remember from school) the graph is a straight line.