MATH 3023- – (Philip Pennance http://pennance.us/) - Absolute value: Summary and exercises

1. Let $x \in \mathbb{R}$. The (non negative) distance between x and the origen is called the *absolute value* of x and is denoted by $|x|$ or $d(x, 0)$

2. Let $x \in \mathbb{R}$

   (a)
   $$|x| = \begin{cases} x; & \text{if } x \geq 0; \\ -x; & \text{if } x < 0 \end{cases}$$

   (b) $|x| = \sqrt{x^2}$

   (c) $|x| = \max\{-x, x\}$

3. Let $x, y \in \mathbb{R}$. The distance $d(x, y)$ between x and y is given by
   $$d(x, y) = |y - x|$$

4. For all $a, b \in \mathbb{R}$ we have:

   (a) $|a| \geq 0$

   (b) $|a| = |-a|$

   (c) $|ab| = |a||b|$

   (d) $|a|^2 = a^2$

5. Let $a > 0$. Then

   (a) $|x| < a \Leftrightarrow -a < x < a$

   (b) $|x| \geq a \Leftrightarrow (x \leq -a \text{ or } x \geq a)$

6. Triangle Inequality
   $$|a + b| \leq |a| + |b| \text{ for all } a, b \in \mathbb{R}$$

Exercises

1. Solve:

   (a) $|x| < 2$

   (b) $|x| > 2$

   (c) $|x| = 2$

   (d) $|x| > -2$

   (e) $|x| < -2$

2. Solve:

   (a) $1 < |x| < 2$

   (b) $|-x| = x$

   (c) $|x + 4| < 2$

   (d) $|x + 4| > 2$

   (e) $|-2x + 1| < 2$

   (f) $|-2x + 1| \geq 2$

   (g) $|-x| = x$

3. The following proof shows that all real numbers are equal. Find the mistake.

   $x, y \in \mathbb{R}$
   Let $m = (x + y)/2$
   $x + y = 2m$
   $(x + y)(x - y) = 2m(x - y)$
   $x^2 - y^2 = 2mx - 2my$
   $x^2 - 2mx = y^2 - 2my$
   $x^2 - 2mx + m^2 = y^2 - 2my + m^2$
   $(x - m)^2 = (y - m)^2$
   $(x - m) = (y - m)$
   $x = y$

4. Find real number a, r with $r > 0$ such that the following statements are equivalent:

   (a) $-7 < x < -1$.

   (b) $|x - a| < r$.

5. Let $x \in \mathbb{R}$. Show that the following statements are equivalent:

   (a) $x^2 \geq |x|$

   (b) $x = 0$ or $x \leq -1$ or $x \geq 1$.

6. Show that the following are not true for all real numbers

   (a) If $x < y$ then $|x| < |y|$.

   (b) $|x| < 3 \Leftrightarrow (x < 3 \text{ or } -x < 3)$

   (c) $|x| > 3 \Leftrightarrow (x > 3 \text{ and } -x > 3)$.