Math 3024 Summary of Lecture I on Complex Numbers

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1. A *complex number* is an ordered pair of real numbers.

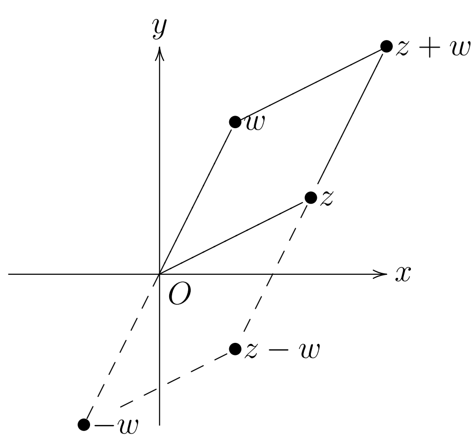
2. The set of complex numbers is denoted by $\mathbb{C}$.

3. If $z = (a, b)$ then a is called the *real* part of z and is denoted Re z. Similarly, b is called the *imaginary* part of z and is denoted Im z.

4. Algebraic operations with complex numbers

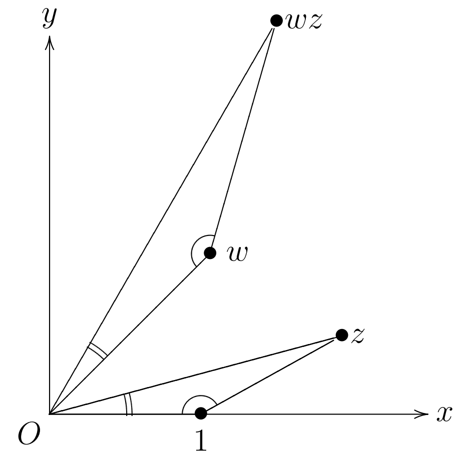
   (a) Addition in $\mathbb{C}$

$$(a, b) + (c, d) = (a + c, b + d)$$



   (b) Multiplication in $\mathbb{C}$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$



   (c) Geometric Interpretation

   Due to the proximity of the next exam, discussion of the geometric interpreations of addition and multiplication will be postponed to the next lecture.

5. Equality of complex numbers: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

6. We often write (x, y) in the "abbreviated" form $x + yi$ as shown in the table:

$(x, y) \in \mathbb{C}$	$x + yi$
$(0, 0)$	0
$(1, 0)$	1
$(2, 0)$	2
$(x, 0)$	x
$(0, 1)$	i
$(0, 2)$	$2i$
$(0, y)$	yi
$(2, -3)$	$2 - 3i$
etc.	etc.

7. Notice that $(0, 1)(0, 1) = (-1, 0)$ or in abreviated form

$$i^2 = -1$$

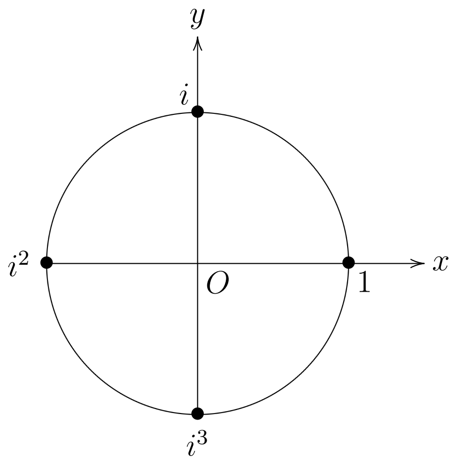
Similary

(a) $i^3 = (0, -1) = -i$

(b) $i^4 = (1, 0) = 1$

(c) $i^5 = (0, 1) = i$

See diagram:



8. Notice that if each number in the (complex) numerical expression $(a, 0) + (b, 0)(0, 1)$ is written in its abbreviated form we obtain $a+bi$ which is the abbreviation of the complex value obtained by evaluating the expression. This is not an accident but follows from the fact that above "abbreviation" defines what mathematicians call a field isomorphism. We omit the details for lack of time. However, it follows that any numerical expression in $\mathbb{C}$ can be correctly evaluated using the abbreviated form together with the relation $i^2 = -1$ and noting that the abbreviations posses the usual field properties of the real numbers.

   (a) Example:

$$\begin{aligned}(a, b) + (c, d) &= (a + ib) + (c + id) \\ &= (a + c) + i(b + d) \\ &= (a + c, b + d)\end{aligned}$$

   (b) Example:

$$\begin{aligned}(a, b)(c, d) &= (a + ib)(c + id) \\ &= ac - bd + i(ad + bd) \\ &= (ac - bd, ad + bc)\end{aligned}$$

9. Let $z \in \mathbb{C}$. The *modulus* of z denoted $|z|$ is the distance of the point z from the origin, i.e., $|z| = d(z, 0)$.

   (a) If $z = (a, b)$ then $|z| = \sqrt{a^2 + b^2}$.

   (b) $|wz| = |w||z|$

   (c) $\left|\frac{w}{z}\right| = \frac{|w|}{|z|}$

   (d) $|w + z| \leq |w| + |z|$

10. Let $z \in \mathbb{C}$. The *conjugate* of z denoted $\overline{z}$ is the reflection of z in the horizontal axis.

   (a) If $z = (a, b)$ then $\overline{z} = (a, -b)$. In abreviated form,

$$\overline{a + bi} = a - bi.$$

   (b) $\overline{w + z} = \overline{w} + \overline{z}$

   (c) $\overline{w - z} = \overline{w} - \overline{z}$

   (d) $\overline{wz} = \overline{wz}$

   (e) Let $c \in \mathbb{R}$. Then $\overline{c} = c$

   (f) $\overline{\left(\frac{w}{z}\right)} = \frac{\overline{w}}{\overline{z}}$

11. Note; Both $z\overline{z}$ and $z + \overline{z}$ are real, i.e., of the form $(x, 0)$. Specifically:

   (a) $z\overline{z} = |z|^2$

   (b) $z + \overline{z} = 2\text{Re}z$

12. Let $z \in \mathbb{C}$ be non zero and define $z^{-1} = \overline{z}|z|^{-2}$, then $zz^{-1} = 1$ so that z^{-1} is the multiplicative inverse of z.

13. Let $z, w \in \mathbb{C}$. Then $\frac{z}{w} = zw^{-1} = z\overline{w}|w|^{-2}$

14. Field Properties of $\mathbb{C}$:

   (a) $(\mathbb{C}, +)$ is an abelian group.

(b) $(\mathbb{C} - 0, \cdot)$ is an abelian group.

Thus $\mathbb{C}$ behaves like $\mathbb{R}$ under the operations of summation and multiplication.

15. Some Graphs:

    (a) $|z| = 2$

    (b) Im $z = 2$

    (c) $|z + 1| = 2$

    (d) $|z + 1| + |z - 1| = 5$

16. Square Roots

    (a) To find the square roots of $z = 3 - 4i$ let $w = a + ib$ and suppose that $w^2 = z$. Then $|w|^2 = |z|$ gives $a^2 + b^2 = 5$. and Re w^2 = Re z yields $a^2 - b^2 = 3$. It follows that $a \in \{2, -2\}$ and $b \in \{1, -1\}$. Since Im w^2 = Im z we have 2ab = -4 so that a and b must have opposite sign. Thus, the only possibilities are $w = (2, -1)$ and $w = (-2, 1)$. In either case it is easy to check that $w^2 = z$.

    (b) If $y > 0$ is real, the two square roots of $-y$ are $i\sqrt{y}$ and $-i\sqrt{y}$

17. Solutions of $az^2 + bz + c = 0$.