Math 3024 Prof. Pennance – Summary of Lecture on Polynomial

1. A definition of a *polynomial* and examples.

2. Notation: Let F be a ring. We denote by $F[x]$ the set of polynomials with coefficients in F. Usually F will be one of the fields $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Example: $\mathbb{Q}[x]$ denotes the polynomials with rational coefficients.

3. The degree of a polynomial. Note that the zero polynomial has degree $-\infty$.

4. Sum and convolution of polynomials.

5. Let $f, g \in F[x]$ where F is a field, then
   (a) $d(f+g) \leq \max\{d(f), d(g)\}$
   (b) $d(f * g) = d(f) + d(g)$

6. Let $F[x]$ denote the set of polynomials over a field F then $(F[x].+, *)$ is a ring, i.e.
   (a) $(F[x], +)$ is an abelian group.
   (b) $\star$ is associative
   (c) $f * (g + h) = f * g + f * h$
   (d) $(g + h) * f = g * f + h * f$

7. Division Theorem
   Let F be a field and $f, g \in F[x]$, with $g \neq 0$. Then there exist unique q and r in $F[x]$ such that $f = qg + r$ and either $r = 0$ or $d(r) < d(g)$ (and so $F[x]$ is Euclidean domain).

8. In the special case $g(x) = x - a$ the division theorem gives
   $$f(x) = q(x)(x - a) + r$$
   and since the degree of r is less than the degree of the divisor, r must be a constant polynomial.

9. Remainder Theorem: Let f be a polynomial over a field F and $a \in F$. Then $f(x) = q(x)(x - a) + f(a)$ and so when the divisor is $x - a$ the remainder in the division theorem is just $f(a)$.

10. Factor Theorem: Let f be a polynomial over a field F and $a \in F$. The following are equivalent:
    (a) $x - a$ is a factor of f.
    (b) $f(a) = 0$.

11. Notice that the factor and remainder theorems are just trivial corollaries of the division theorem.

12. If
    $$p(x) = a_0x^3 + a_1x^2 + a_2x + a_3$$
    and $\alpha \in F$, then
    $$p(\alpha) = ((a_0\alpha + a_1)\alpha + a_2)\alpha + a_3$$

13. Generalizing the previous example we obtain the synthetic substitution algorithm to evaluate a polynomial of degree n.
    $$p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$
    Algorithm: Synthetic Substitution
    Input: $a_0,\ a_1,\ a_2,\ \ldots,\ a_n\ \alpha$
    Initialization: $q_0 := a_0$;
    Iteration : $q_i := \alpha q_{i-1} + a_i$, $1 \leq i \leq n$.
    Output: $q_0, q_1, \ldots, q_n$

14. Claim: Let $q_0, q_1, \ldots, q_n$ as above and
    $$q(x) = q_0x^{n-1} + q_1x^{n-2} + \cdots, +q_{n-1}$$
    and $r = q_n$, then
    $$p(x) = (x - \alpha)q(x) + r$$
    and so the synthetic substitution algorithm determines both the quotient and remainder when $p(x)$ is divided by $x - \alpha$.

15. Comparison of the running times for the synthetic substitution algorithm and direct evaluation.

16. Example: Divide $p(x) = 2x^3 - 9x^2 + 14x - 8$ by $x - 3$.
Solution: Apply the the synthetic substitution algorithm with $\alpha = 3$.
$q_0 = 2$
$q_1 = 3(2) - 9 = -3$
$q_2 = 3(-3) + 14 = 5$
$q_3 = 3(5) - 8 = 7$
Hence $p(3) = 7$ and
$$\frac{2x^3 - 9x^2 + 14x - 8}{x - 3} = 2x^2 - 3x + 5 + \frac{7}{x - 3}$$

17. Show $x + 1$ is a factor of $x^3 + x^2 + x + 1$

18. Show using synthetic division that $x - 17$ is a factor of $x^3 - 6x^2 - 177x - 170$

19. Find k given that $x + 1$ is a factor of $x^5 + (3k - 1)x^4 - 5x^3 + (k + 2)x^2 + 1$.

20. Lemma: If a is a factor of bc and $\text{gcf}(a, b) = 1$ then a is a factor of c.

21. Rational root theorem.
Let $a_0x^n + a_1x^{n-1} + \cdots a_n \in \mathbb{Z}[x]$. If $h/k \in \mathbb{Q}$ is a root and h, k have no common factor (other than 1), then the numerator h is a factor of the constant term a_n and the denominator k divides the leading coefficient a_0.

Proof. $n = 1$. Let $p(x) = a_0x + a_1$. Suppose $p(\frac{h}{k}) = 0$, then $a_0\frac{h}{k} + a_1 = 0$ and so $a_0h + a_1k = 0$. From this we see that h is a factor of a_1k. Since $\text{gcf}(h, k) = 1$. it follows from the preceeding lemma that h is a factor of a_1. Now $a_0h + a_1k = 0$ also implies that k is a factor of a_0h and in this case the lemma shows that k must be a factor of a_0.

Proof. $n = 2$. See class notes.

22. Application. There does not exist $\alpha \in \mathbb{Q}$ such that $\alpha^2 - 2$. (i.e. there is no rational root of 2).

23. Given that 2 is a root of
$$p(x) = x^4 - 4x^3 + x^2 + 12x - 12$$
Find all rational roots of p.