Math 3024 Prof. Pennance – Summary of Lecture II on Polynomials

1. An *invertible* element of a ring is one possessing a multiplicative inverse in the ring.

2. Example: The only invertible elements in $\mathbb{Z}$ are 1 and -1.

3. Let $n \in \mathbb{Z}$ be non zero. A factorization $n = ab$, where $a, b \in \mathbb{Z}$ is called *trivial* if either a or b is invertible.

4. Suppose $n \in \mathbb{Z}$ is non zero and not invertible. Then n is called *irreducible* if n is has only trivial factorizations; i.e., n is *irreducible* if $n = ab$ where $a, b \in \mathbb{Z}$ implies a is invertible or b is invertible.

5. $7 = ab$ implies $a \in \{1, -1\}$ or $b \in \{1, -1\}$ and so 7 is irreducible.

6. A non-zero polynomial f over a field F is invertible if and only if f is constant.

7. Let $f \in F[x]$ be non zero and non invertible. The f is called *irreducible* if $f = ab$ and $a, b \in F[x]$ imply a is invertible or b is invertible.

8. Let f be a polynomial over a field F. If f has a factor $a_0x + a_1$ of degree 1, then f has a factor of the form $x - \alpha$. For example if $3x + 4$ is a factor of a polynomial over $\mathbb{R}$, then so is $(x - \frac{4}{3})$.

9. Claim: $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$.
   Proof: Suppose $x^2 - 2 = ab$ where $a, b \in \mathbb{Q}[x]$. If a is of degree 1, we can, by the preceeding observation, assume that $a(x) = x - \alpha$. By the factor theorem α must then be a rational root of $x^2 - 2$ which is impossible. Thus either a has degree 2 or degree zero. In either case, one of a or b must be a constant non zero polynomial and hence invertible.

10. Examples:
    (a) $x^2 - 2$ is not irreducible in $\mathbb{R}[x]$.
    (b) $x^2 + 2$ is irreducible in $\mathbb{R}[x]$.
    (c) $x^2 + 2$ is reducible in $\mathbb{C}[x]$.
    (d) $x - \alpha$ is irreducible in $\mathbb{C}[x]$.

11. Fundamental Theorem of Algebra *Every polynomial in $\mathbb{C}[x]$ has a complex root.*
    It follows that if $f \in \mathbb{C}[x]$ have degree n, then f has a factorization of the form
    $$f(x) = c(x - w_1)(x - w_2) \cdots (x - w_n).$$
    In the case $n = 2$ we can find the factorization by completing the square as in the following example:
    $$\begin{aligned} x^2 - 4x + 5 &= (x-2)^2 + 1 \\ &= (x-2)^2 - i^2 \\ &= (x - 2 - i)(x - 2 + i) \end{aligned}$$
    Remark. Notice that the factorization has the form $(x - z)(x - \bar{z})$ where $z = 2 + i$. This is **not** an accident as the next result shows.

12. Let $f \in \mathbb{R}[x]$. Then
    $$f(z) = 0 \Rightarrow f(\bar{z}) = 0.$$
    It follows that if $(x - z)$ is a factor of f then so is $(x - \bar{z})$.

13. If $z \in \mathbb{C}$ then
    $$(x - z)(x - \bar{z}) = x^2 - (2\text{Re } z)x + |z|^2.$$
    Example. Noting that $\text{Re}(2 + i) = 2$ and $|2 + i|^2 = 5$ we have
    $[x - (2 + i)][x - (2 - i)] = x^2 - 4x + 5.$

14. Exercises:
    (a) Find b if one solution of $x^2 + bx + 1 = 0$ is twice the other.

(b) If $x^2 + bx + c$ has roots r, s show that $rs = c$ and $r + s = -b$.

(c) Find a polynomial $p \in \mathbb{R}[x]$ of possible lowest possible degree such that $1/3$ and $1 + i$ are roots.

(d) Show that $x^2 - 2$ has no rational roots.

(e) Factor $x^4 - x^2 - 2$ completely over each field:

i. $\mathbb{Q}$ ii. $\mathbb{R}$ iii. $\mathbb{C}$

(f) Example: Given that 2 is a root of

$$p(x) = x^4 - 4x^3 + x^2 + 12x - 12$$

Find all roots of p.