Math 4995 Prof. Pennance – Notes on Digraphs II

1. Let $(V, \lesssim)$ be a preorder. For every pair $a, b \in V$ define

$$a \sim b \text{ if } a \lesssim b \text{ and } b \lesssim a.$$

Then $\sim$ is an equivalence relation on V.

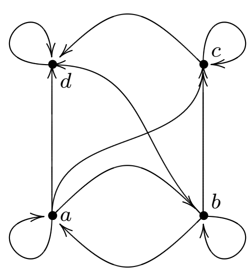
2. Let $[x]$ and $[y]$ be equivalence classes of $\sim$. Define $[x] \lesssim_c [y]$ if $x \lesssim y$. Then:

   (a) $\lesssim_c$ is well defined.

   (b) $\lesssim_c$ is a preorder relation on the set of classes of $\sim$, (reflexivity and transitiviy being inherited from $\lesssim$).

   (c) $\lesssim_c$ is actually an order.
   Proof of antisymmetry:
   If $[x] \lesssim_c [y]$ and $[y] \lesssim_c [x]$ then $x \lesssim y$ and $y \lesssim x$ so we have $x \sim y$. Since $\sim$ is an equivalence relation it follows that $[x] = [y]$.

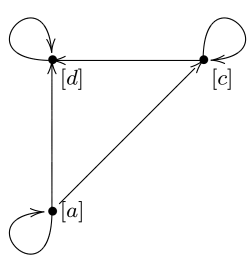
3. Example. If $G = (V, E)$ be the preorder drawn below.



The set of classes of the relation $\sim$ is

$$V/\sim \ = \{\{a\}, \{b, c\}, \{d\}\}.$$

The ordered digraph $(V/\sim, \lesssim_c)$ is drawn below.



4. Let $(V, \lesssim)$ a preorder and let A be a subset of V. An element $a \in A$ is called:

   (a) a *minimal* element of A if there does not exist $a' \in A$ such that $a' < a$.

   (b) a *maximal* element of A if there does not exist $a' \in A$ such that $a' > a$.

   (c) a *minimum* element of A if for all $x \in A \setminus a$ we have $a < x$.

   (d) a *maximum* element of A if for all $x \in A \setminus a$ we have $a > x$.

5. Let $(V, \lesssim)$ a preorder and let A be a subset of V. An element $v \in V$ is called:

   (a) a lower bound of A if $v \lesssim x$ for all $x \in A$.

   (b) an upper bound for A if $x \lesssim v$ for all $x \in A$.

   (c) a greatest lower bound (glb) of A if v is a lower bound of A and in addition $v' \lesssim v$ for every other lower bound v' of A.

   (d) a least upper bound (lub) of A if v is an upper bound of A and in addition $v \lesssim v'$ for every other upper bound v' of A.

6. Remarks.

   (a) A lower and upper bounds of a subset A are not necessarily elements of A

   (b) Minimal, maximal, minimum, and maximum elements of A must be elements of A.

   (c) In an ordered set $(V, \leq)$, a set A can have at most one glb and at most one lub. These are denoted glb A and lub A respectively.