Math 4995 – Exercises on Dirichlet's pigeon-hole principle

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1. A box contains 700 socks; 300 red, 200 blue and 200 green. How many socks must be selected to guarantee a matching pair.

2. Show that every rational number has a finite or repeating decimal expansion.

3. Five points are selected from an equilateral triangle of sides 2. Show that 2 of the five points are on a distance at most 1.

4. Show that in a simple undirected graph without loops there exist two vertices of the same degree.

5. Prove that the product of any n consecutive positive integers is divisible by $n!$

6. An element of $\mathbb{Z} \times \mathbb{Z}$ is called a lattice point. Show that among any 5 lattice points there are two whose midpoint is also a lattice point.

7. Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \bigcup B$. If $d(v) = 1$ for all $v \in A$ and $|A| > q|B|$ for some $q \in \mathbb{N}$ show that there exists $v \in B$ such that $d(v) > q$.

8. In Ramsey's game two players each with a different color take turns to color an edge of the complete graph k_6. The first player to color a triangle in his color loses. Show that one of the players must always lose.

9. The edges of the complete graph K_{17} are colored in three colors. Show that there is at least one triangle which has three sides of the same color.

10. Let f be a permutation. Show that there is an integer n such that f^n is the identity permutation.

11. Let A and B be finite sets. If $f : A \to B$ is surjective, show that $A \geq B$.

12. Show that among any five distinct points in an equilateral triangle of side 2 there are two points at distance at most 1.

13. If one hen lays one egg in one day, how many eggs do one and one half hens lay in one and one half days.

14. A hen lays at least one egg each day during a period of 30 days. Her average is no more than 1.5 eggs per day. Show that there is a period of consecutive days during which she lays exactly 14 eggs.

15. Show that a group of order 4 is abelian.

16. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ Suppose that 5 numbers from A are selected. Show that the selection contains at least one pair whose sum is 9.

17. Let $f : A \to B$ where A and B are finite and $|A| = |B|$. Is it true that if f is not injective, then f is not surjective? Prove or disprove.

18. Let $f : A \to A$ be an injection. Show that if A is finite then f is also a bijection.

19. From $\{1, 2, \ldots 2n\}$, $n + 1$ numbers are selected. Show that the selection contains two numbers which are relatively prime.