MATH 3XXX- Lecture Notes
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Summary – Exponential Functions

Definition Let $a > 0$, $a \neq 1$ be a fixed real number. The function $\mathbb{R} \to (0; +\infty)$ given by $x \mapsto a^x$ is called the *exponential function* of *base a*.

Recall that a^x can be defined (see Dolciani) as the limit of the sequence a^{x_n} where x_n is any sequence of rationals converging to x. It can be shown that this limit exists and that the resulting function $x \mapsto a^x$ is continuous. We omit the proof of this fact since it will proven in the calculus course. The following properties. of the exponential function follow directly from properties of the limit and corresponding properties of rational exponents.

Theorem

1. Let $a > 1$ then $a^x > 1$ for all $x > 0$.
2. $a^x a^y = a^{x+y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$
5. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
6. $\frac{a^x}{a^y} = a^{x-y}$

Proof See Dolciani and your class notes.

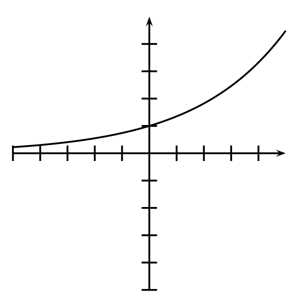
Theorem $a^x > 0$ for all $x \in \mathbb{R}$.

Proof

For any integer n, $a^x = \dfrac{a^{x+n}}{a^n}$. Pick n so that $x + n > 0$. Then if $a > 1$ the right hand side is positive by property (1) above. On the other hand, if $0 < a < 1$ then $\dfrac{1}{a} > 1$ and so $a^x = \left(\dfrac{1}{a}\right)^{-x} > 0$ by the first case.

Theorem If $a > 1$ the exponential function with base a is *strictly increasing* (i.e. if $x < y$ then $a^x < a^y$).

Proof If $x < y$, by property (2) above, $a^y - a^x = a^x(a^{y-x} - 1)$. By property (1), the right hand side is positive and so $a^x < a^y$.



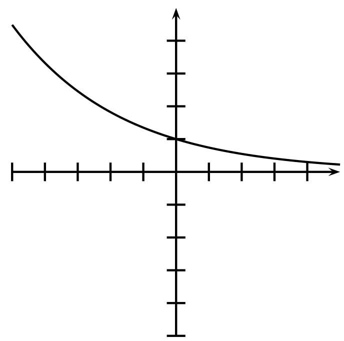
$x \mapsto a^x$
$a > 1$

The drawing above depicts the graph of a typical exponential function with base $a > 1$. It will be proven in the calculus course that the graph is concave up and that the horizontal axis is an asymptote. It can also be shown that the function takes every value in the interval $(0, \infty)$.

Theorem If $0 < a < 1$ the exponential function with base a is *strictly decreasing* (i.e. if $x < y$ then $a^x > a^y$).

Proof Exercise (Hint: $a^x = \left(\frac{1}{a}\right)^{-x}$)

The drawing below depicts the graph of a typical exponential function with base $0 < a < 1$.

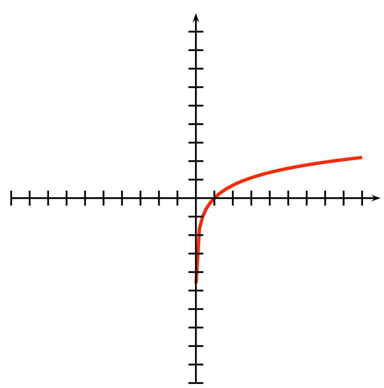


$x \mapsto a^x$
$0 < a < 1$

Theorem The exponential function is injective.

Proof Suppose $x \neq y$. Without loss of generality assume $x < y$, Since an exponential is either strictly increasing or decreasing we must have that either $a^x < a^y$ or $a^x > a^y$. In either case $a^x \neq a^y$. This shows that $x \mapsto a^x$ is injective.

Corollary The exponential function $x \mapsto a^x$ has an inverse function. This function is denoted $\log_a x$.



$x \mapsto \log_a x$
$a > 1$

Since the exponential function defines a bijection between $\mathbb{R}$ and its image $(0, \infty)$ the domain of $x \mapsto \log_a x$ is $(0, \infty)$ and its image $\mathbb{R}$. We therefore have:

Theorem

1. $\log_a(a^x) = x$ for all $x \in \mathbb{R}$.
2. $a^{\log_a x} = x$ for all $x > 0$.