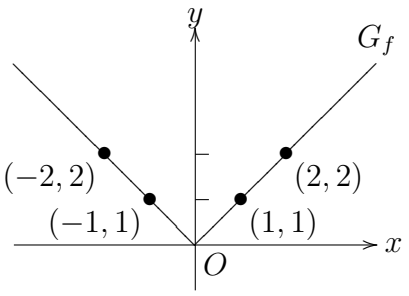
1. The function $x \mapsto |x|, \quad x \in \mathbb{R}$ whose graph (sketched below) is given by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0; \end{cases}$$

is called the *absolute value* function.



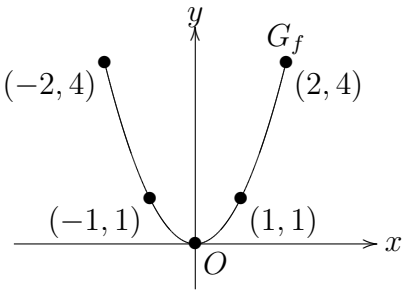
2. Let F be a point the plane and D a line not containing F. Let $d(P, D)$ denote the perpendicular distance from P to D. The set

$$\{P \in \mathbb{R}^2 : d(P, F) = d(P, D)\}$$

is called the *parabola* with *focus* F and *directrix* D.

3. The graph of the function given by $f(x) = x^2, \quad x \in \mathbb{R}$ is a parabola.

Proof. Let D be the line with equation $y = -1/4$, F the point $(0, 1/4)$, and, $P = (x, y)$. By the distance formula (exercise), the condition d(P,F) = d(P, D) is equivalent to $x^2 = y$. This shows that the graph of f is the the parabola with focus $(0, 1/4)$ and directrix $y = -1/4$.



4. Let F be a point the plane, D a line not containing F and $\epsilon > 1$. The set $\{P \in \mathbb{R}^2 : d(P, F) = \epsilon d(P, D)\}$ is called the *hyperbola* with *focus* F *directrix* D. The constant ϵ is called the *eccentricity*.

5. The graph of $f(x) = 1/x, \quad x \in \mathbb{R} - 0$ is the hyperbola H with directriz D the line $y = -x + \sqrt{2}$, focus $F = (\sqrt{2}, \sqrt{2})$ and, eccentricity $\epsilon = \sqrt{2}$.

Proof. Let $P = (x, y) \in H$. Then, (exercise),

$$d^2(P, D) = \frac{1}{2}(x + y - \sqrt{2})$$

By the distance formula

$$d^2(P, F) = (x - \sqrt{2})^2 + (y - \sqrt{2})^2$$

It easily follows that the condition

$$d(P, F) = \sqrt{2}\ d(P, D)$$

is equivalent to

$$y = 1/x, \quad x \in \mathbb{R} - 0$$

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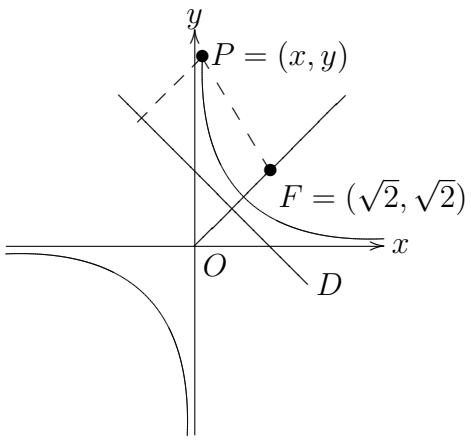
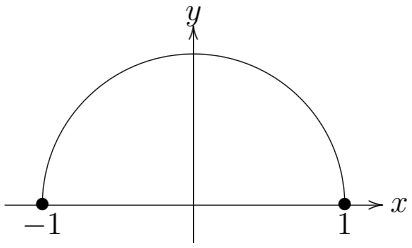


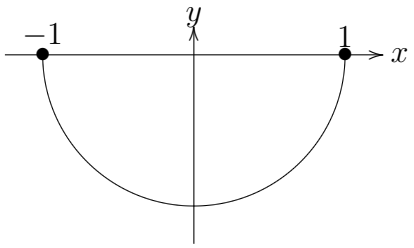
6. The graph of the function $f(x) = \sqrt{1-x^2}$, with domain $[-1, 1]$ is a semicircle.

   Proof. Let S be the unit circle with equation $x^2 + y^2 = 1$. Let $(x, y) \in G_f$, then $y = \sqrt{1-x^2}$ which implies $x^2 + y^2 = 1$. This shows that $G_f \subseteq S$ But $\sqrt{1-x^2} \geq 0$ so G_f must be contained in the semicircle which is the intersection of the upper half plane with S. Moreover, since every non negative number has a principal square root, the graph G_f must consist of all points on this semicircle.

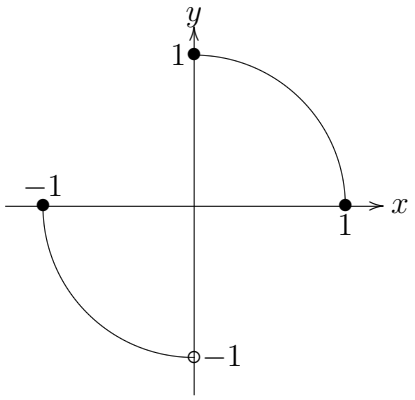


7. An infinite number of functions satisfy the equation $x^2 + y^2 = 1$ Let f be any such function. Let $(x, y) \in G_f$. Since $x^2 + y^2 = 1$ the point (x, y) must lie on the unit circle S. Hence $G_f \subseteq S$. There are many subsets of the unit circle which are the graph of a function. Three such are shown below.

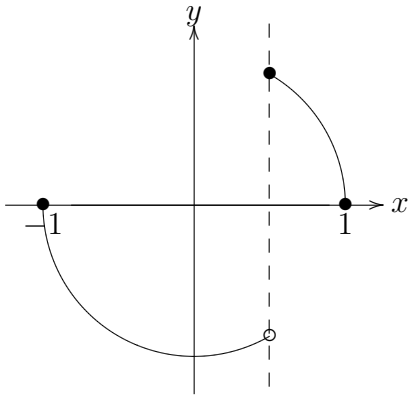
   (a)



   (b)



   (c)



   Each of these functions is said to be *implicitly* defined by the equation $x^2 + y^2 = 1$.

Exercises

1. Show that the parabola with focus $F = (0, p)$ and D the line $y = -p$ is the graph of the function $f(x) = \dfrac{1}{4p}x^2$.

2. Find the focus and directriz of the parabola $y = ax^2$.

3. Show that the expresion

$$f(x) = \begin{cases} x, & \text{if } x > 0; \\ -x, & \text{if } x < 1; \end{cases}$$

does not define a function.

4. Show that

$$\{(m/n, m) : m, n \in \mathbb{Q}, n \neq 0\}$$

is not the graph of a function.

5. How many relations are there with domain $\{1, 2, 3, 4\}$ and codomain $\{a, b, c\}$.? How many functions?.

6. The conditions $f(n) = 2f(n-1)$, $f(0) = 1$ define the graph of a function with domain $\mathbb{N}$. Find $f(10)$. Given that $f(a) = 2048$) find $f(a-3)$.

7. Attempting to define $f(x)$ in terms of $f(x)$ does not generally define a function. Show that the expression

$$f(x) = \begin{cases} 1, & \text{if } x = 1; \\ 1 + f(x/2), & \text{if } x \text{ is even}; \\ f(3x-1), & \text{if } x \text{ is odd}; \end{cases}$$

does not define the graph of a function with domain $\mathbb{N}$.

8. Verify that the graph of the function $f(x) = \sqrt{x}, \quad x \geq 0$ is one half of a parabola. Find the focus and directrix of this parabola.

9. A function $f : \mathbb{R} \to \mathbb{R}$ is called *linear* if for all real numbers x and y and constant c.

   L1. $f(x + y) = f(x) + f(y)$

   L2. $f(cx) = cf(x)$

   (a) Let m be a constant. Show that a function of the form $f(x) = mx$ is linear.

   (b) Show that the function $f(x) = mx + b$ with $b \neq 0$ is not linear.

   (c) Show that the absolute value function is not linear.