

## Finding the image of a function

Let  $f : A \rightarrow B$ . Recall that the image of  $f$  is the set

$$\text{im } f = \{f(a) : a \in A\}.$$

It follows that  $b \in \text{im } f$ , if and only if  $\exists a \in A$  such that  $f(a) = b$ . Therefore, to check that a given element  $b \in B$  is in  $\text{im } f$  it suffices to show that the equation:  $f(x) = b$  has a solution  $x \in A$ . The image of  $f$  is just the set of  $b \in B$  such that  $f(x) = b$  has a solution. Finding this set may be easy or very hard depending on the nature of the function  $f$ . We illustrate with some simple examples.

**Example** Let  $f : \{x \in \mathbb{R} : x \geq 0\} \rightarrow \mathbb{R}$ , be given by  $f(x) = \sqrt{x}$ . Find  $\text{im } f$ .

**Solution** The equation  $f(x) = b$  is equivalent to  $\sqrt{x} = b$ ,  $x \geq 0$ . We are required to find the set of all elements  $b$  in the codomain  $\mathbb{R}$  for which this equation has a solution. If  $b < 0$  there is no solution since, by definition, the principal square root is non negative. If  $b \geq 0$  then  $\sqrt{b^2} = |b| = b$  and so  $x = b^2$  is a solution of the equation  $f(x) = b$ . Thus  $\text{im } f = \{x \in \mathbb{R} : x \geq 0\}$ .

**Example** Let  $f : \{x \in \mathbb{R} : x \geq 0\} \rightarrow \mathbb{R}$  be given by  $f(x) = \sqrt{x} + 7$ . Find  $\text{im } f$ .

**Solution** The equation:  $f(x) = b$  is equivalent to  $\sqrt{x} + 7 = b$ ,  $x \geq 0$ . We must find the set of all elements  $b$  in the codomain  $\mathbb{R}$  for which this equation has a solution. If  $b \geq 7$ , then  $b - 7 \geq 0$ . In this case:

$$\begin{aligned} \sqrt{(b-7)^2} + 7 &= |b-7| + 7 \\ &= b - 7 + 7 \\ &= b \end{aligned}$$

Therefore  $x = (b - 7)^2$  is a solution  $f(x) = b$ . On the other hand if  $b < 7$ , the equation  $f(x) = b$  has no solution since  $\sqrt{(b-7)^2}$  is non negative by definition. We conclude that  $\text{im } f = \{b \in \mathbb{R} : b \geq 7\}$ .

**Example** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = \sqrt{x^2 + 1}$ , find  $\text{im } f$ .

**Solution** The equation  $f(x) = b$  is equivalent to  $\sqrt{x^2 + 1} = b$ ,  $x \in \mathbb{R}$ . If  $b < 0$  this clearly has no solutions. On the other hand if  $b \geq 0$

$$\begin{aligned} \sqrt{x^2 + 1} &= b \\ \Rightarrow x^2 + 1 &= b^2 \\ \Rightarrow x^2 &= b^2 - 1 \end{aligned}$$

Clearly if  $b^2 < 1$  there is no solution. If  $b^2 \geq 1$  then

$$x^2 = b^2 - 1 \Rightarrow x = \sqrt{b^2 - 1} \text{ or } x = -\sqrt{b^2 - 1}$$

Substitution reveals that both of these are solutions. Thus  $f(x) = b$  has solutions if and only if  $b \geq 0$  and  $b^2 \geq 1$ . But  $b \geq 0$  and  $b^2 \geq 1$  is equivalent to  $b \geq 1$ . Hence  $\text{im } f = \{b \in \mathbb{R} : b \geq 1\}$ .

**Example** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = -3 - |2x|$ . Find  $\text{im } f$ .

**Solution**

$$\begin{aligned} f(x) &= b \\ \Leftrightarrow -3 - |2x| &= b, \quad x \in \mathbb{R} \\ \Leftrightarrow |2x| &= -(b + 3) \end{aligned}$$

Clearly if  $b > -3$  this can have no solutions. On the other hand, if  $b \leq -3$ , then  $x = \frac{b+3}{2}$  is a solution since

$$\begin{aligned} f(x) &= -3 - |2\frac{b+3}{2}| \\ &= -3 - |b + 3| \\ &= (-3) + (b + 3) \quad (\text{since } b + 3 \leq 0) \\ &= b \end{aligned}$$

Hence  $\text{im } f = \{b \in \mathbb{R} : b \leq -3\}$ .