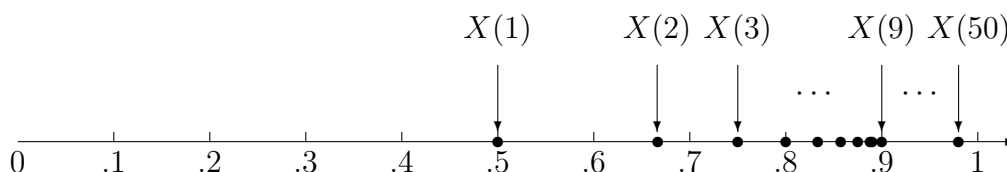


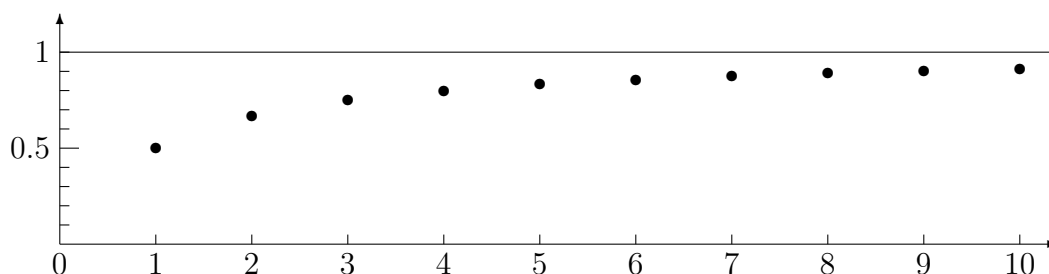
The Limit of a Sequence

Consideration of the behaviour of a sequence $X(n)$ for arbitrarily large n leads to the concept of the limit of a sequence as n tends to infinity. Our aim is to make this notion mathematically precise. We first consider a simple example.

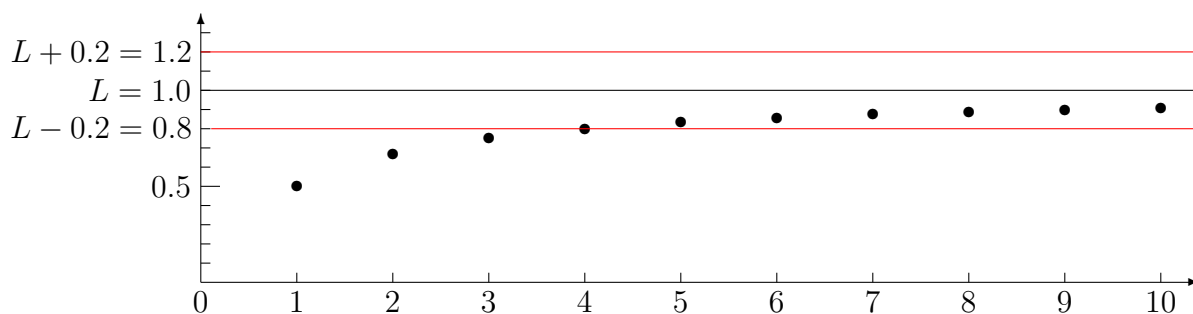
Let $X : \mathbb{N} \rightarrow \mathbb{R}$ be the sequence given by $X(n) = \frac{n}{n+1}$ the first few terms of which are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$. We can examine this sequence by drawing a few terms $X(n)$ on the number line



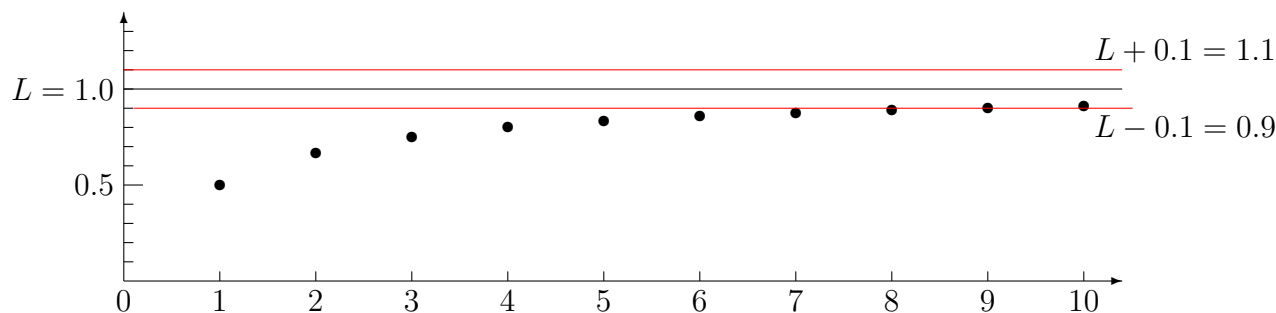
or by drawing a graph of $X(n)$ versus n in the plane:



Since for any natural number n , the quantity $\frac{n}{n+1} < 1$. It follows that $X(n)$ is bounded above by 1. Furthermore if $n < m$ then (exercise) $\frac{n}{n+1} < \frac{m}{m+1}$ and so $X(n)$ is an increasing function. Moreover, the distance between $X(n)$ and 1 is $|X(n) - 1| = \frac{1}{n+1}$. We can make this distance as small as we wish by choosing n large enough. For example, if $n = 4$ the distance between $X(n)$ and 1 is $\frac{1}{4+1} = 0.2$ and for all $n > 4$ this distance is strictly less than 0.2. This means that for all $n > 4$ we have that $|X(n) - 1| < 0.2$ as illustrated in the figure below. Thus, for all $n > 4$, the points of the graph of $X(n)$ lie in the horizontal strip bounded by the horizontal lines through the points $(0, 0.8)$ and $(0, 1.2)$ shown in red.



In the same way, if $n = 9$ the distance between $X(n)$ and 1 is $\frac{1}{9+1} = 0.1$ and since $X(n)$ is increasing and bounded above by 1 we have that for all $n > 9$ the distance between $X(n)$ and 1 is strictly less than 0.1. Thus $n > 9$ implies that $|X(n) - 1| < 0.1$.



More generally, we claim that given any number $\epsilon > 0$ (no matter how small), we can find a real number $N > 0$ such that $|X(n) - 1| < \epsilon$ for all $n > N$. The above calculations show that if $\epsilon = 0.2$ then $N = 4$ works and if $\epsilon = 0.1$ then $N = 9$ or indeed any number larger than 9 suffices. It is clear from the examples that the required value of N depends on the size of ϵ . For an arbitrary value of ϵ , if $N = \frac{1}{\epsilon}$ then $n > N$ implies $n > \frac{1}{\epsilon}$ from which it follows that $\frac{1}{n+1} < \epsilon$. In conclusion, given $\epsilon > 0$ we have found, (by working backwards from the desired result), a number N such that $|X(n) - 1| < \epsilon$ whenever $n > N$. This makes precise, the intuitive statement that the terms of the sequence approach the number 1 in the limit as n tends to infinity. We formalize this in a definition.

Definition A number L is called the *limit* of a sequence $X : \mathbb{N} \rightarrow \mathbb{R}$ if for every $\epsilon > 0$, there exists a real number N such that $|X(n) - L| < \epsilon$ whenever $n > N$.

If a sequence $X(n)$ has a limit L we write $\lim_{n \rightarrow \infty} X(n) = L$. We also write $X(n) \rightarrow L$ as $n \rightarrow \infty$ and say that “ X has limit L as n tends to infinity”. A sequence X is *convergent* if it has a limit. Otherwise it is called *divergent*.

Example From the discussion preceding the above definition it follows that $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$

and so the sequence is convergent.

Example Find the limit of the sequence $X(n) = 1/n$.

Solution The first few terms are $1, 1/2, 1/3, 1/4$, etc. and a graph can be easily drawn as in the example above. Intuitively it should be “clear” that the limit is 0. However (since many statements which appear “clear” are in fact wrong), we must prove this fact using the definition of limit. Therefore, Let $\epsilon > 0$. We must find N such that $|1/n - 0| < \epsilon$ whenever $n > N$. By working backwards it is easily to see guess that $N = 1/\epsilon$ will have the desired property. To check this, notice that $n > N$ implies $n > 1/\epsilon$ which in turn implies $|1/n - 0| < \epsilon$ as required. In conclusion, we have found $N > 0$ such that $|1/n - 0| < \epsilon$ for

all $n > N$. By definition $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

To understand this procedure, suppose you wish to make the difference between $1/n$ and 0 is less than $\epsilon = .001 = 1/1000$. Choose $N = 1/\epsilon = 1000$. Then if $n > N$, $1/n < 1/1000$.

Example Find the limit of the constant sequence $X(n) = 5$.

Solution We claim that the limit of X is 5. Let $\epsilon > 0$. This time it is easy to find N such that $|X(n) - 5| < \epsilon$ whenever $n > N$ Just take $N = 24$ (any other number will do !). Since $|X(n) - 5| = 0$ for all $n \in \mathbb{N}$ we have in particular that $|X(n) - 5| < \epsilon$ whenever $n > 24$. Thus $\lim_{n \rightarrow \infty} X(n) = 5$.

More generally if $X(n) = c$ for all $n \in \mathbb{N}$ then $\lim_{n \rightarrow \infty} X(n) = c$.

Exercise Show that if a sequence has a limit then the limit is unique. Hint: Use contradiction.

Proving the existence of N such that $|X(n) - L| < \epsilon$ whenever $n > N$ is not always as simple as in the previous example. However the following algebraic properties of limits simplify their calculation.

Theorem

If $\lim_{n \rightarrow \infty} X(n) = L$ and $\lim_{n \rightarrow \infty} Y(n) = M$. Then:

1. $\lim_{n \rightarrow \infty} (X(n) \pm Y(n)) = L \pm M$
2. $\lim_{n \rightarrow \infty} (X(n) \cdot Y(n)) = L \cdot M$
3. $\lim_{n \rightarrow \infty} \frac{X(n)}{Y(n)} = \frac{L}{M}$ if $M \neq 0$ and $Y(n) \neq 0$ for all n .

Proof See Dolciani.

Example Find the limit of the sequence $a : \mathbb{N} \rightarrow \mathbb{R}$ given by $a(n) = 3/n$.

Solution Notice that $a(n) = X(n) \cdot Y(n)$ where $X(n) = 1/n$ and $Y(n) = 3$ for all $n \in \mathbb{N}$. Now $\lim_{n \rightarrow \infty} X(n) = 0$ and $\lim_{n \rightarrow \infty} Y(n) = 3$ It follows from part (2) of the above theorem that $\lim_{n \rightarrow \infty} a(n) = 0 \cdot 3 = 0$

Example Find the limit of the sequence given by $a(n) = \frac{1}{n^2}$.

Solution Let $X(n) = 1/n$. Then $a(n) = X(n) \cdot X(n)$. We have already shown that $\lim_{n \rightarrow \infty} X(n) = 0$. By the theorem

$$\lim_{n \rightarrow \infty} a(n) = \lim_{n \rightarrow \infty} X(n) \cdot \lim_{n \rightarrow \infty} X(n) = 0 \cdot 0 = 0$$

By induction we can generalize this example and show that $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ for any natural number $p > 0$

Example Find the limit of the sequence $X(n) = \frac{n^2 + 3n}{3 + n^2}$.

Solution $X(n) = \frac{1 + \frac{3n}{n^2}}{\frac{3}{n^2} + \frac{n^2}{n^2}} = \frac{1 + \frac{3}{n}}{\frac{3}{n^2} + 1}$

Now

$$\lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right) = \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{3}{n} = 1 + 0 = 1$$

Also

$$\lim_{n \rightarrow \infty} \left(\frac{3}{n^2} + 1\right) = \lim_{n \rightarrow \infty} \frac{3}{n^2} + \lim_{n \rightarrow \infty} 1 = 0 + 1 = 1$$

Hence

$$\lim_{n \rightarrow \infty} X(n) = \frac{\lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right)}{\lim_{n \rightarrow \infty} \left(\frac{3}{n^2} + 1\right)} = 1$$

Definition A sequence $X : \mathbb{N} \rightarrow \mathbb{R}$ is *bounded* if there exists a number M such that $|X(n)| \leq M$ for all $n \in \mathbb{N}$. A sequence which is not bounded is called *unbounded*

Example Consider the sequence $X(n) = \frac{n}{n+1}$. Since $|X(n)| \leq 1$ for all $n \in \mathbb{N}$, the sequence is bounded.

Theorem A convergent sequence is bounded.

Proof Let X be a convergent sequence with limit L . From the definition of a limit there exists a number N such that $|X(n) - L| < 1$ for all $n > N$. It follows (exercise) that all terms of X are less than or equal in absolute value to the largest element M of the set $\{X(1), X(2), \dots, X(N), L + 1, L - 1\}$

This result is very useful for showing that a given sequence has no limit. For example, the sequence $X(n) = n$ has no limit since it is unbounded. Note however that the converse is false. A counterexample is provided by the sequence $X(n) = (-1)^n$ which is bounded by 1 yet does not converge.