Math 3023 – Prof. Pennance – Supplementary notes on lines

Vertical Lines

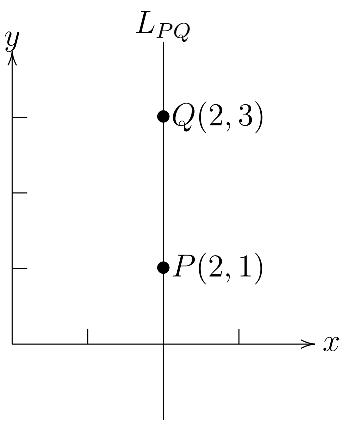
Let $P = (x_1, y_1)$, and $Q = (x_2, y_2)$ be distinct points in the plane with $x_1 = x_2 = c$. The set

$$L_{PQ} = \{(x, y) \in \mathbb{R}^2 : x = c\}$$

is called the *(vertical) line determined by the points P and Q.*

Example

The drawing below shows the vertical line L_{PQ} with $P = (2, 1)$ and $Q = (2, 3)$ The equation of L_{PQ} is $x = 2$.

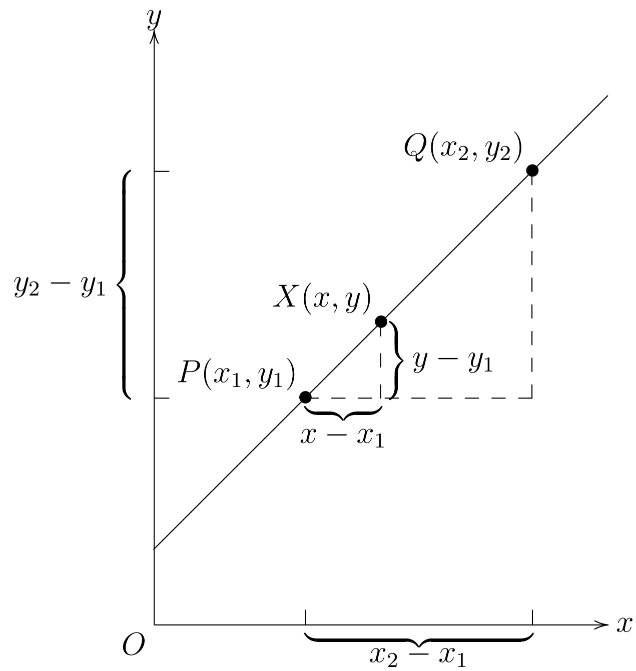


Trivial Exercises Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points with $x_1 = x_2 = c$. Show:

1. $P,\ Q \in L_{PQ}$.
2. $L_{QP} = L_{PQ}$.
3. If $P, Q \in L_{AB}$, where A and B are distinct points, then $L_{AB} = L_{PQ}$.
4. L_{PQ} is the unique vertical line containing P and Q.

Non vertical lines

Definition Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points with $x_2 \neq x_1$. We define the *slope determined by the ordered pair* (P, Q) to be the number $m_{PQ} = \dfrac{y_2 - y_1}{x_2 - x_1}$



Exercise. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points with $x_2 \neq x_1$. Show that $m_{PQ} = m_{QP}$.

Definition. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points with $x_2 \neq x_1$. Let m denote the common value of m_{PQ} and m_{QP}. The set

$$\begin{aligned} L_{PQ} &= \{X \in \mathbb{R}^2 : m_{PX} = m_{PQ}\} \cup P \\ &= \{(x, y) \in \mathbb{R}^2 : \frac{y - y_1}{x - x_1} = m\} \cup \{(x_1, y_1)\} \end{aligned}$$

is called the *line determined by the ordered pair P and Q*.

Exercises. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points with $x_2 \neq x_1$. Show:

1. $L_{PQ} = \{(x, y) \in \mathbb{R}^2 : y - y_1 = m(x - x_1)\}$
   (The equation $y - y_1 = m(x - x_1)$ is called the *point slope form* of the line.)

2. $P,\ Q \in L_{PQ}$.

Claim. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points with $x_2 \neq x_1$. Then $L_{PQ} = L_{QP}$

Proof. Let $(x, y) \in L_{PQ}$. Then $y - y_1 = m_{PQ}(x - x_1)$. Since $Q \in L_{PQ}$ we have $y_2 - y_1 = m(x_2 - x_1)$. It follows that $(y - y_1) + (y_1 - y_2) = m(x - x_1) - m(x_2 - x_1)$ and so $y - y_2 = m(x - x_2)$. This means that $(x, y) \in L_{QP}$ and therefore we have proven that $L_{PQ} \subseteq L_{QP}$. A similar argument shows that $L_{QP} \subseteq L_{PQ}$. Therefore $L_{PQ} = L_{QP}$

Claim. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be distinct points with $x_2 \neq x_1$. Let $A \in L_{PQ}$. Then $L_{PA} = L_{PQ}$

Proof. Let $A = (x, y)$. Since $A \in L_{PQ}$ we have by definition that $m_{PA} = m_{PQ} = m$. Therefore

$$\begin{aligned} A \in L_{PQ} &\Leftrightarrow y - y_1 = m_{PQ}(x - x_1) \\ &\Leftrightarrow y - y_1 = m_{PA}(x - x_1) \\ &\Leftrightarrow A \in L_{PA} \end{aligned}$$

Hence $L_{PA} = L_{PQ}$.

Claim. Let P, Q distinct points in the plane. If $P, Q \in L_{AB}$ where A, B are distinct points then $L_{AB} = L_{PQ}$

Proof. The case where L_{PQ} is vertical is exercise (3) above. Therefore we may suppose that P and Q do not lie on the same vertical line vertical line. Since $P \in L_{AB}$ we have (by the previous claim) that $L_{PB} = L_{AB}$. Similarly, since $Q \in L_{AB}$ we have $L_{QA} = L_{AB}$. Therefore $L_{QA} = L_{PB} \ni P$. From $P \in L_{QA}$ we have $L_{PQ} = L_{QA}$. It follows that $L_{PQ} = L_{AB}$.

Corollary. Let P, Q be distinct points in the plane. Then L_{PQ} is the only line containing both P and Q.