MATH 3023- Prof. Pennance - Order: Summary and exercises

Order

1. We asume the existence of a subset $\mathbb{R}^{\geq} \subseteq \mathbb{R}$ called the set of *non-negative* real numbers possessing the following properties which we use as axioms:

   O1. If $x, y \in \mathbb{R}^{\geq}$ then $x + y \in \mathbb{R}^{\geq}$ and $xy \in \mathbb{R}^{\geq}$. (*closure*)

   O2. For all $x \in \mathbb{R}$, either $x \in \mathbb{R}^{\geq}$ or $-x \in \mathbb{R}^{\geq}$.

   O3. If $x \in \mathbb{R}^{\geq}$ and $-x \in \mathbb{R}^{\geq}$ then $x = 0$.

2. Let $x, y \in \mathbb{R}$. We say that x is *less than or equal* to y written $x \leq y$ if $y - x \in \mathbb{R}^{\geq}$. In this case we also say that y is *greater than or equal* to x and write $y \geq x$. (Thus $y \geq x$ is equivalent to $x \leq y$)

3. **Claim** $0 \in \mathbb{R}^{\geq}$

   Proof
   By O2. either $0 \in \mathbb{R}^{\geq}$ or $-0 \in \mathbb{R}^{\geq}$. But $-0 = 0$.

4. **Claim** The binary relation $\leq$ is an *order*. i.e., for all $x, y, z \in \mathbb{R}$ the following are true:

   (a) $x \leq x$ (*reflexivity*).

   (b) $x \leq y$ and $y \leq x$ then $x = y$ (*antisymmetry*)

   (c) ($x \leq y$ and $y \leq z$) $\Rightarrow x \leq z$ (*transitivity.*).

   Proof

   (a) Reflexivity. $x - x = 0 \in \mathbb{R}^{\geq}$. Hence $x \leq x$.

   (b) Antisymmetry. Suppose $x \leq y$ and $y \leq x$. Then $y - x \in \mathbb{R}^{\geq}$ and $x - y \in \mathbb{R}^{\geq}$. But $y - x = -(x - y)$ so by O3. $x - y = 0$. Therefore $x = y$.

   (c) Transitivity. Suppose $x \leq y$ and $y \leq z$. Then $y - x \in \mathbb{R}^{\geq}$ and $z - y \in \mathbb{R}^{\geq}$. Hence by the closure property O1. $(z - y) + (y - x) \in \mathbb{R}^{\geq}$. Therefore $z - x \in \mathbb{R}^{\geq}$ which means $x \leq z$.

5. Let $x, y \in \mathbb{R}$. We say that x is *less than* y written $x < y$ if $x \leq y$ and $x \neq y$. The set of positive real numbers is defined to be the set $\mathbb{R}^{+} = \{x \in \mathbb{R} : 0 < x\}$.

6. Let $a, b \in \mathbb{R}$ with $a < b$. The following sets are called *intervals*.

   (a) $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
   (b) $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
   (c) $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
   (d) $(a, \infty) = \{x \in \mathbb{R} : a < x\}$
   (e) $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$
   (f) $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$
   (g) $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$

   Notice that a square bracket is indicates that the corresponding endpoint is included in the interval.

Exercises

1. Let $x, y \in \mathbb{R}$. Prove the following:

   (a) If $x \leq y$ and $c \leq 0$ then $cx \geq cy$.
   (b) If $x \leq y$ and $0 \geq c$ then $cx \geq cy$.
   (c) If $x \leq y$ and $z \in \mathbb{R}$ then $x + z \leq y + z$.
   (d) If $x, y \in \mathbb{R}^{\geq}$ and $x \leq y$ then $x^2 \leq y^2$
   (e) $0 \leq x^2$
   (f) If $x \leq x$ then $4xy \leq (x + y)^2$

(g) If $x, y \in \mathbb{R}^{\geq}$ then $\sqrt{xy} \leq \frac{x+y}{2}$. (This result is called the *arithmetic–geometric mean inequality*)

2. Let $a, b, c, d \in \mathbb{R}$. Suppose that $a \leq b$ and $c \leq d$. Which of the following are true. Prove or disprove your answer.

   (a) $ac \leq bd$.

   (b) $a - c \leq b - d$.

   (c) $a/c \leq b/d$.

   (d) $a + c \leq b + d$.

3. Let $A = \{x \in \mathbb{R} : x < 3\}$ and $B = \{x \in \mathbb{R} : x < 4\}$. Let $\bar{A} = \mathbb{R} \setminus A$ and $\bar{B} = \mathbb{R} \setminus B$ be the complements of A and B. Sketch each of the following sets on the number line or say if it is empty.

   (a) $A \cap B$

   (b) $A \cap \bar{B}$

   (c) $A \cup B$

   (d) $A \cup \bar{B}$

   (e) $\bar{A} \cap B$

   (f) $\bar{A} \cap \bar{B}$

   (g) $\bar{A} \cup B$

   (h) $\bar{A} \cup \bar{B}$

4. Find real number a, r with $r > 0$ such that the following statements are equivalent:

   (a) $-7 < x < -1$.

   (b) $|x - a| < r$.

5. Show that **NONE** of the following statements are true for all real numbers. For each statement find the set of real numbers for which it is true.

   (a) $x \geq -x$

   (b) $-x \leq 0$

   (c) If $x < 2$ then $x^2 < 4$

   (d) If $1/x \geq 1$ then $x \leq 1$

   (e) $\sqrt{x^2} = x$ and $\sqrt{x^2} = -x$

   (f) $\sqrt{x^2} = -x$

6. Solve

   (a) $1/x > 1$

   (b) $1/x > 0$

   (c) $x^4(x - 1) > 0$

   (d) $x^2 > x^3$

   (e) $|x|^2 = x^2$

7. Let x, y be real numbers with $x > y$. Show that $x^3 > y^3$.

8. Let a, b, c be natural numbers.
   If $1 \leq a < b < c$ show that

$$\frac{bc + ac + ab}{abc} < \frac{11}{6}.$$

9. Find a real number $\delta > 0$ such that whenever $0 < x < \delta$, we have $1/x > 100$ and prove that the real number you found has the required property. Say what you can about the uniquness of δ?

10. Find an upper bound for $\frac{1}{x^2 + 2x^4 + 4}$.