Math 3023 - Prof. Pennance – On predicates and quantifiers

1. Quantifiers
   (a) $\forall$ "for all"
   (b) $\exists$ "there exist"

2. Quantifiers are used in conjunction with variables to form statements such as:
   (a) "$\forall x \in \{2, 7, 13\}$, x is prime" –a true statement equivalent to the conjunction "2 is prime and 7 is prime and 13 is prime".
   (b) "$\exists y \in \{1, 4, 8\}$ such that y is prime" –a false statement equivalent to the disjunction: "1 is prime or 4 is prime or 8 is prime".

3. A variable x which is preceded by $\forall x$ or $\exists x$ is called *bound* or *quantified.* A variable which is not bound is called *free.*

4. Examples
   (a) The variable x in the expression "x is prime" is free.
   (b) The variable x in the expression "$\forall x \in \{4, 8, 12\}$ x is prime" is bound.
   (c) The variable x in the expression "$\forall y \in \{4, 8, 12\}$ x is prime" is free.

5. An expression which contains a free variable is called a *predicate.*

6. Examples of predicates
   (a) x is prime
   (b) x is dishonest
   (c) $y^2 \geq 0$

7. Given a predicate $P(x)$ with free variable x, we denote by $P(a)$ the result of substituting a for x.

8. A predicate $P(x)$ with variable x is said to be *valid* for a set A if for any $a \in A$, $P(a)$ is a statement (i.e., has a truth value).

9. Example
   The predicate $P(x)$, "x is dishonest", is valid for the set of politicians in the Senate of Puerto Rico. The reader can substitute his "favorite" politician for the free variable and reflect upon the truth or falsehood of the resulting statement. However, $P(x)$ is not valid for the set $\mathbb{N}$ of natural numbers since expressions such as $P(5)$, "5 is dishonest", do not have truth values.

10. It follows from the preceding definitions that a predicate $P(x)$, valid for a set A, can be converted to a statement by:
    (a) substitution of values from A,
    (b) quantification over A (or a subset of A).

11. Examples
    Let $P(x)$ be the predicate $x < 2$ which is valid for $\mathbb{N}$ so we can substitute values from $\mathbb{N}$, for example:
    (a) $P(1)$ is the true statement $1 < 2$.
    (b) $P(2)$ is the false statement $2 < 2$.

    We can also quantify over subsets of $\mathbb{N}$. For example:
    (a) $\exists x \in \{1, 4, 8\}$ $x < 2$
    (b) $\forall x \in \{1, 4, 8\}$ $x < 2$

    are both statements. The first is true and the second is false.