MATH 3023 – Some Properties of the Real Numbers

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1. Cancelation Theorems for Addition
   For all $a, b, c \in \mathbb{R}$.

   (a) if $a + c = b + c$ then $a = b$.
   (Right cancelation)

   (b) if $c + a = c + b$ then $a = b$.
   (Left cancelation)

   Proof (a) Suppose $a + c = b + c$. Since c has an additive inverse $-c$ it follows that $(a + c) + (-c) = (b + c) + (-c)$. Associativity gives $a + (c + (-c)) = b + (c + (-c))$. By the property of the additive inverse the latter implies $a + 0 = b + 0$. Finally by the property of the additive identity 0 we have $a = b$, which was to be proven. The proof of (b) is similar.

2. The additive inverse of any real number x is unique.

   Proof Suppose x has two additive inverses, a and b. We claim that $a = b$. Since a and b are additive inverses of x we have $a + x = 0$ and $x + b = 0$. Hence $a + x = b + x$. By the cancellation theorem $a = b$.

3. For all $a \in \mathbb{R}$, $-(-a) = a$.

   Proof By the property of additive inverse, $-a$ has an inverse $-(-a)$ such that $-a + [-(-a)] = 0$. Since $-a$ is the inverse of a, it satisfies $(-a) + a = 0$. Therefore $(-a) + [-(-a)] = (-a) + a$. By the cancellation theorem $-(-a) = a$.

4. For all $x \in \mathbb{R}$, $0 \cdot x = 0$

   Proof Since 0 is the additive inverse. $0x + 0 = 0x = (0 + 0)x$. By the distributive law $(0 + 0)x = 0x + 0x$. By transitivity of equality, $0x + 0 = 0x + 0x$. Using the cancellation theorem it follows that $0x = 0$.

5. For all $x \in \mathbb{R}$, $-x = -1 \cdot x$

   Proof Using respectivly the property of the multiplicative inverse, the distributive law, and the property of the additive inverse, we obtain $x + (-1) \cdot x = 1 \cdot x + (-1) \cdot x = (1 + (-1)) \cdot x = 0.x$. It follows (using the previous theorem) that $x + (-1) \cdot x = 0$. Since the additive inverse is unique we conclude that $-1 \cdot x = -x$.

6. For all $a, b \in \mathbb{R}$, $(-a) \cdot b = -(a \cdot b)$

   Proof Using the previous theorem and the associative law
   $-(a \cdot b) = -1 \cdot (a \cdot b) = (-1 \cdot a) \cdot b = (-a) \cdot (b)$

7. For all $a, b \in \mathbb{R}$,

   $$ab = 0 \Rightarrow (a = 0 \text{ or } b = 0)$$

   Proof Suppose that $ab = 0$. If $a = 0$ the conclusion is true. If $a \neq 0$ then a has a unique multiplicative inverse a^{-1}. It follows that $a^{-1}(ab) = a^{-1}0$. By [4] above the right hand side of this equation is zero. By associativity and the definition of multiplicative inverse, the left hand side $a^{-1}(ab) = (a^{-1}a)b = 1b = b$. We conclude that in this case $b = 0$. In both possible cases we have proven that $(a = 0 \text{ or } b = 0)$ as required.

8. Application. Solve $x^2 = 4$.

   Solution $x^2 = 4 \Rightarrow x^2 - 2^2 = 0$ and so $(x - 2)(x + 2) = 0$. By [7] either $x - 2 = 0$ or $x + 2 = 0$. Hence either $x = 2$ or $x = -2$. Substitution verifies that both values are indeed solutions.

Exercises

1. Use the properties of the real numbers solve the following equations over $\mathbb{R}$.

   (a) $2x + 3 = 7$.

   (b) $ax + b = c$

   (c) $x^2 - 2x - 63 = 0$

   (d) $x^2 - 5 = 0$.
   (You may assume that 5 has a principal square root.)

2. Find the error in the following argument:

   Let $a = 1$ and $b = 1$ Then $a = b$ and $a^2 = ab$. It follows that $a^2 - b^2 = ab - b^2$ from which $(a + b)(a - b) = b(a - b)$ After canceling the common factor $a - b$ we find $a + b = b$ Putting $a = b = 1$ gives $2 = 1$

3. Show that the following are true for all real numbers a, b, c, d.

   (a) If $c \neq 0$ and $ac = bc$, then $a = b$. (Cancellation property for multiplication)

   (b) $a - b = -(b - a)$

   (c) $(a^{-1})^{-1} = a, \quad a \neq 0$

   (d) $-(ab) = (-a)b = a(-b)$

   (e) $\dfrac{-a}{b} = -\dfrac{a}{b} = \dfrac{a}{-b}, \quad b \neq 0.$

   (f) $(ab)^{-1} = b^{-1}a^{-1}, \quad a, b \neq 0$

   (g) $\dfrac{ab}{c} = \dfrac{a}{b}c$

   (h) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}, \quad b, d \neq 0$

   (i) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}, \quad b, d \neq 0$

   (j) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b}\dfrac{d}{c}, \quad b, c, d \neq 0$

4. Show that the following statements are false:

   (a) $\dfrac{x^2 - 1}{x + 1} = x - 1$ for all $x \in \mathbb{R}$

   (b) $\sqrt{x^2} = x$, for all $x \in \mathbb{R}$

   (c) $\exists y \in \mathbb{R} : y^2 - 3y + 4 = 0$

   (d) $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x + y = 3$

   (e) $\forall x \in \mathbb{R}, \ |-x| = x$

   (f) $x^2 > 16 \Rightarrow [(x > 4 \text{ or } x > -4)]$ for all $x \in \mathbb{R}$