Math 4021 – Prof. Pennance – On Substitution

1. Let X a set variables, F a formula and $q : X \to \text{Term}$ a "term assignment". Our aim is twofold.

   (a) To define $F(q)$ – the formula obtained by "substituting" each $x \in V_f(F) \cap X$ by $q(x)$.

   (b) To define carefully when this substitution makes sense.

2. Definition.

   (a) If c is a constant $c(q) = c$

   (b) If x is a variable

$$x(q) = \begin{cases} q(x), & \text{if } x \in X; \\ x, & \text{if } x \notin X; \end{cases}$$

3. $F(q)$ is now defined recursively as follows:

   (a) $(t_1 = t_2)(q) \;=\; t_1(q) = t_2(q)$

   (b) $rt_1 \ldots t_n(q) \;=\; rt_1(q) \ldots t_n(q)$ where r is a relational symbol of order n.

   (c) $(\neg F)(q) = \neg F(q)$

   (d) $(F \to G)(q) = F(q) \to G(q)$

   (e) $(\forall x F)(q) = \forall x F(q')$ where q' is the restriction of q to $X' = X - x$

4. If $q(x_1) = t_1, q(x_2) = t_2, \ldots, q(x_n) = t_n$ then $F(q)$ is often denoted $F\left[\frac{x_1 \ldots x_n}{t_1 \ldots t_n}\right]$

   In the special case $n = 1$, in which a single variable x is substituted by a term t, definition (3) reads:

   (a) If F atomic then $F\left[\frac{x}{t}\right]$ is obtained from F by replacing x by t.

   (b) $(\neg F)\left[\frac{x}{t}\right] = \neg F\left[\frac{x}{t}\right]$

   (c) $(F \to G)\left[\frac{x}{t}\right] = F\left[\frac{x}{t}\right] \to G\left[\frac{x}{t}\right]$

   (d) $(\forall x F)\left[\frac{y}{t}\right] = \begin{cases} \forall x F & \text{if } y = x; \\ \forall x F\left[\frac{y}{t}\right] & \text{if } y \neq x; \end{cases}$

5. It follows from (3) that only free variables are subject to change under a substitution. For example,

$$(\forall x\; x{+}y = y{+}x)\left[\frac{x}{z}\right] = \forall x\; x{+}y = y{+}x$$

6. Warning. If we fail to restrict q and use instead the INCORRECT formula

$$(\forall x F)(q) = \forall x F(q)$$

   we obtain the INCORRECT result

$$(\forall x\; x{+}y = y{+}x)\left[\frac{x}{z}\right] = \forall x\; z{+}y = y{+}z$$

   which does not have the intended meaning.

7. There is still a problem to be overcome. For example, using the correct substitution rule (3e) we have

$$(\forall x\; x{+}y = y{+}x)\left[\frac{y}{x}\right] = \forall x\; x{+}x = x{+}x$$

   which is incorrect semantically. The problem arises because, even though y is free, it falls within the scope of the quantifier $\forall x$. It follows that any term t containing the variable x, (in the above example $t = x$) will be subject to quantification upon substitution for y. We must therefore limit ourselves to substitutions which are "proper" for logical deduction. The notation of "proper" is made precise by the following definition.

8. Let q be a term assignment and F a formula. We define the relation $S(q, F)$ read q is a *proper substitution* in F as follows:

(a) If F is atomic, then $S(q, F)$.

(b) If $S(q, F)$ then $S(q, \neg F)$.

(c) If $S(q, F)$ and $S(q, G)$ then $S(q, F \rightarrow G)$

(d) $S(q, \forall x F)$ if

   i. $S(q', F)$ and

   ii. For all $v \in V_f[\forall x F \cap X]$, $x \notin V(q(v))$.
   (recall, V(t) denotes the set of variables of a term t)

   Note: (ii) can be written

$$x \notin V(q[V_f(\forall x F)])$$

   Thus a substitution q in $\forall x F$ is improper if either q' is improper in F, or, for some free variable $v \in \forall x F$, $q(v) = t$ where t contains the quantified variable x.

9. Examples

(a)

$$(\exists x\ x + x = y) \left[\frac{y}{x}\right] = \exists x + x = x$$

is improper since $V(q(y)) = \{x\}$ and x is quantified.

(b)

$$(\forall x\ x + y = y + x) \left[\frac{x}{y}\right]$$

is proper. In this case, the domain of q' is empty and no free variables is eligible for substitution by a term. Thus, the substitution yields

$$\forall x\ x + y = y + x$$

(c)

$$(\forall y\ \forall x\ x + y = y + x) \left[\frac{x}{y}\right]$$

equals

$$\forall y (\forall x\ x + y = y + x) \left[\frac{x}{y}\right]$$

which by the previous example equals

$$\forall y \forall x\ x + y = y + x$$

Notice that the substitution leaves the formula inchanged. It is left as an exercise to show that the substitutin is proper.